Overview

Complex Filtering and Integration via Sampling

- Signal processing
  - Sample then filter (remove aliases) then resample
  - Nonuniform sampling: jittering and Poisson disk
- Statistics
  - Monte Carlo integration and probability theory
  - Sampling from distributions
  - Sampling from shapes
  - Sequential, adaptive and stratified sampling

Cameras

$$R = \int_T \iint_\Omega \iint_A L(x,\omega,t)\, P(x) S(t) \cos\theta \, dA \, d\omega \, dt$$

Motion Blur

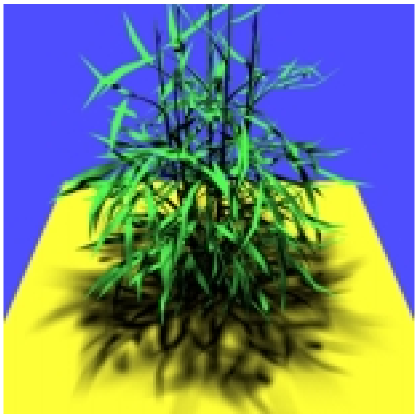
Source: Cook, Porter, Carpenter, 1984

Depth of FIeld

Source: Mitchell, 1991

Lighting and Soft Shadows

$$E(x) = \int_{H^2} L(x', x) V(x', x) \cos\theta d\omega$$



Visibility

Rendering Operators

The measurement equation

$$R = \int_T \iint_\Omega \iint_A L(x, \omega, t)\, P(x) S(t) \cos\theta\, dA\, d\omega\, dt$$

Lighting equation

$$E(x) = \int_{H^2} L(x', x) V(x', x) \cos\theta d\omega$$

Lots of darn integrals:

- **Integrate over *t*: Motion blur**
- **Integrate over pixel *x*, *y*: Antialiasing**
- **Integrate over lens *u*, *v*: Depth of field**
- **Integrate over ω (A): Penumbras**
- **Integrate over paths with *k* bounces: Light transport**

Random Variables

Random variables

- X **is chosen by some random process**
- $X_i \sim p(x)$ **probability distribution**
- $Y_i = f(X_i)$ **is also a random variable**

Expected values

$$E[f] \equiv \int f(x)\,p(x)\,dx$$

Monte Carlo Integration

Definite integral $I = \int_0^1 f(x)\,dx$

Expectation of *f* $E[f] = \int_0^1 f(x)\,p(x)\,dx$

Random variables $X_i \sim p(x)$

$Y_i = f(X_i)$

Estimator $F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$

Unbiased $E[F_N] = I(f)$

Unbiased Estimator

$$E[F_N] = I(f)$$

Properties

$$E[f] \equiv \int f(x)\, p(x)\, dx$$

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i] = \frac{1}{N}\sum_{i=1}^{N} E[f(X_i)]$$

$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x)\, p(x)\, dx$$

$$= \frac{1}{N}\sum_{i=1}^{N} \int_0^1 f(x) dx$$

$$= \int_0^1 f(x)\, dx$$

Assume uniform probability distribution for now

Monte Carlo Algorithms

Advantages

- **Easy to implement**
- **Easy to think about (but be careful of biasing!)**
- **Robust when used with complex integrands and domains (shapes, lights, …)**
- **Efficient for high dimensional integrals**
- **Efficient solution method for a few selected points**

Disadvantages

- **Slow (many samples needed for convergence)**
- **Noisy**

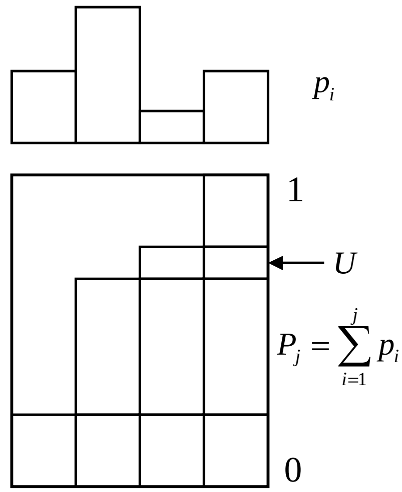
Discrete Probability Distributions

Random variables chosen by some random process

Discrete events X_i with probability p_i

$$p_i \geq 0$$

$$\sum_{i=1}^{n} p_i = 1$$



Construction of samples

To randomly select an event,

Select X_i if

$$p_1 + \cdots p_{i-1} < U \leq p_1 + \cdots p_{i-1} + p_i$$

Uniform random variable

Examples

Sampling a set of point light sources

Associate a probability with each light

$$p_i = \frac{\Phi_i}{\sum_{j=1}^{n} \Phi_j}$$

Continuous Probability Distributions

Cumulative probability distribution function

$$P(x) = \Pr(X_i < x)$$

$$\Pr(\alpha \le X_i \le \beta) = \int_\alpha^\beta p(x)\,dx = P(\beta) - P(\alpha)$$

Probability density function

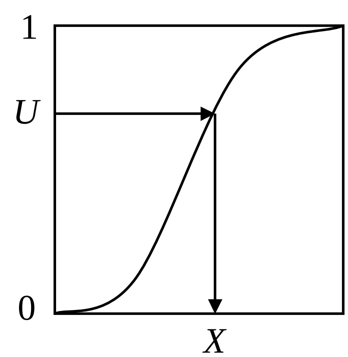
$$p(x) = \frac{dP(x)}{dx}$$

Construction of samples

Solve for $X = P^{-1}(U)$

Must know:

1. **The integral of $p(x)$**
2. **The inverse function $P^{-1}(x)$**



Example: Power Function

Assume

$$p(x) = (n+1)x^n$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) = \sqrt[n+1]{U}$$

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

Trick: $Y = \max(U_1, U_2, \cdots, U_n, U_{n+1})$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U < x) = x^{n+1}$$

Sampling a Circle

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2}\right)\Bigg|_0^1 \theta\Big|_0^{2\pi} = \pi$$

$$p(r,\theta)\,dr\,d\theta = \frac{1}{A} r\,dr\,d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}$$

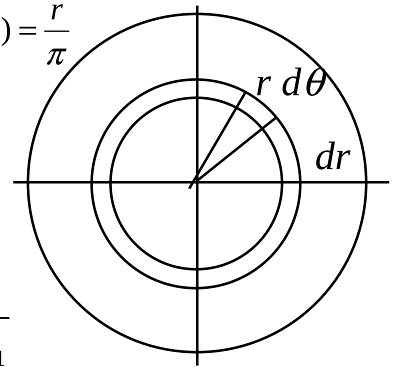
$$p(r,\theta) = p(r)\,p(\theta)$$

$$p(r) = r$$

$$P(r) = \frac{1}{2} r^2$$

$$p(\theta) = 1$$

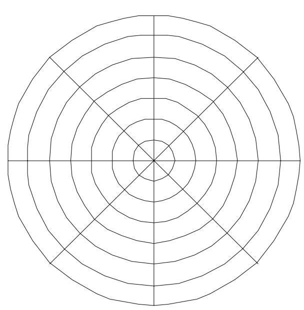
$$P(\theta) = \frac{1}{2\pi}\theta$$



$$r = \sqrt{U_1}$$
$$\theta = 2\pi U_2$$

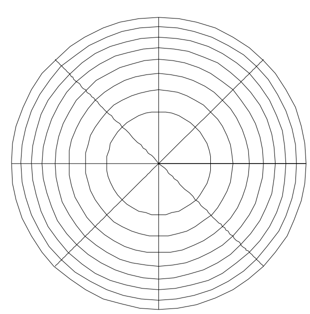
Sampling a Circle

WRONG! <> Equi-Areal



$$r = U_1$$
$$\theta = 2\pi U_2$$

RIGHT = Equi-Areal

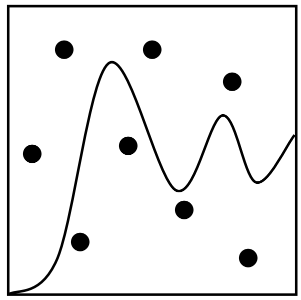


$$r = \sqrt{U_1}$$
$$\theta = 2\pi U_2$$

Rejection Methods

$$I = \int_0^1 f(x)\,dx$$

$$= \iint_{y<f(x)} dx\,dy$$



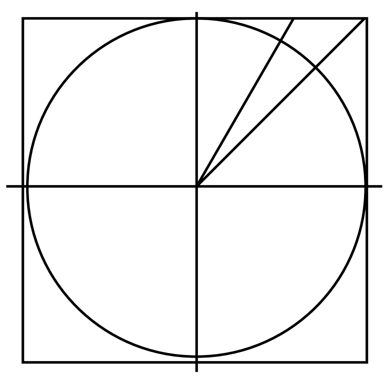
Algorithm

Pick U_1 and U_2

Accept U_1 if $U_2 < f(U_1)$

Wasteful? Efficiency = Area / Area of rectangle

Sampling a Circle: Rejection



do {

$X=1-2U_1$

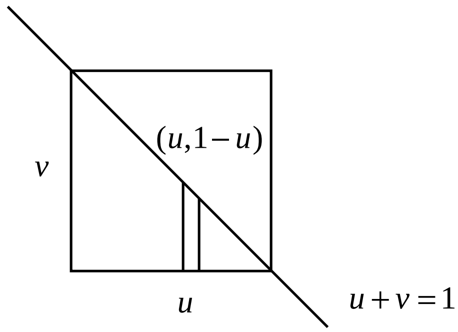
$Y=1-2U_2$

while($X^2+ Y^2 >1$)

May be used to pick random 2D directions
Circle techniques may also be applied to the sphere

Sampling a Triangle

$u \geq 0$
$v \geq 0$
$u + v \leq 1$



$$A = \int_0^1 \int_0^{1-u} dv\, du = \int_0^1 (1-u)\, du = -\frac{(1-u)^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$

CS348B Lecture 8 — Pat Hanrahan, Spring 2001

Sampling a Triangle

Here *u* and *v* are not independent!

Conditional probability

$$p(u) \equiv \int p(u,v)\, dv \qquad p(u \mid v) \equiv \frac{p(u,v)}{p(u)}$$

Triangle

$$p(u,v) = 2$$

$$p(u) = 2 \int_0^{1-u} dv = 2(1-u)$$

$$p(v \mid u) = \frac{1}{(1-u)}$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\, du \quad = (1-u_0)^2$$

$$P(v_0 \mid u_0) = \int_0^{v_o} p(v \mid u_0)\, dv \quad = \int_0^{v_o} \frac{1}{(1-u_0)} dv \quad = \frac{v_0}{(1-u_0)}$$

$$u_0 = 1 - \sqrt{U_1}$$
$$v_0 = \sqrt{U_1} U_2$$

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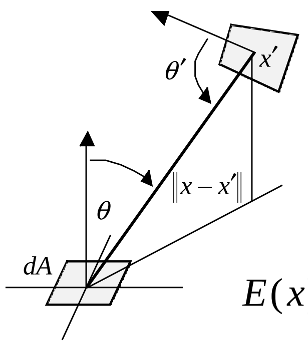
Direct Lighting



$$E(x) = \int_{H^2} L_s(x', x) V(x', x) \cos\theta \, d\omega$$

$$\cos\theta \, d\omega = \frac{\cos\theta' dA'}{\|x - x'\|^2}$$

$$E(x) = \int_A L_s(x', x) V(x', x) \frac{\cos\theta \cos\theta'}{|x - x'|^2} dA'$$

$$= \frac{1}{N} \sum_i L_s(x_i', x) V(x_i', x) \frac{\cos\theta \cos\theta_i'}{|x - x_i'|^2} dA_i'$$

Convert from an solid angle integral to an area integral

Variance

Definition

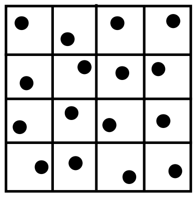
$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2 - 2YE[Y] + E[Y]^2]$$
$$= E[Y^2] - E[Y]^2$$

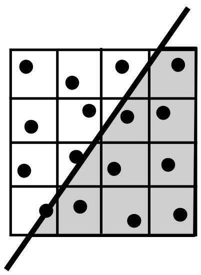
Properties

$$V[\sum_i Y_i] = \sum_i V[Y_i]$$

$$V[aY] = a^2 V[Y]$$

Stratified Sampling





Stratified sampling like jittered sampling

Allocate samples per region

$$N = \sum_{i=1}^{m} N_i \qquad F_N = \frac{1}{N}\sum_{i=1}^{m} N_i F_i$$

New variance $V[F_N] = \frac{1}{N^2}\sum_{i=1}^{N} N_i V[F_i]$

Thus, if the variance in regions is less than the overall variance, there will be a reduction in resulting variance

For example: An edge through a pixel

$$V[F_N] = \frac{1}{N^2}\sum_{i=1}^{\sqrt{N}} V[F_j] = \frac{V[F_R]}{N^{1.5}}$$

High-dimensional Sampling

Stratified sampling (also numerical quadrature)

For a given error …

$$E \sim \frac{1}{N^d}$$

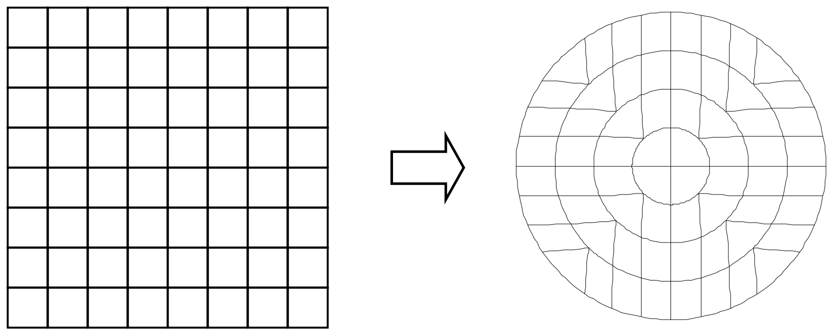
Random sampling

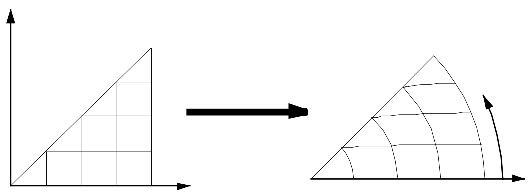
For a given variance …

$$E \sim V^{1/2} \sim \frac{1}{N^{1/2}}$$

Monte Carlo much better for integration in high dimensional spaces

Shirley's Mapping





$$r = U_1$$

$$\theta = \frac{\pi}{4}\frac{U_2}{U_1}$$

Convergence

Mean and standard deviation

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

$$E[\mu_N] = E[\frac{1}{N}\sum_i Y_i] = \frac{1}{N}\sum_i E[Y_i] = E[Y]$$

$$V[\mu_N] = V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N}V[Y]$$

$$\sigma_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \mu_N)^2$$

Sequential Sampling

Central Limit Theorem

$$\lim_{N \to \infty} \Pr\left\{ \mu_N - E[Y] \le \frac{t\sigma}{N^{1/2}} \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx$$

Sample rays until confidence in the estimate is high

- **Student t-distribution**

  Purgathofer

- **Chi-squared distribution**

  Lee, Redner, Uselton (1985)

Essentially the procedures used by pollsters

Adaptive Sampling

Whitted

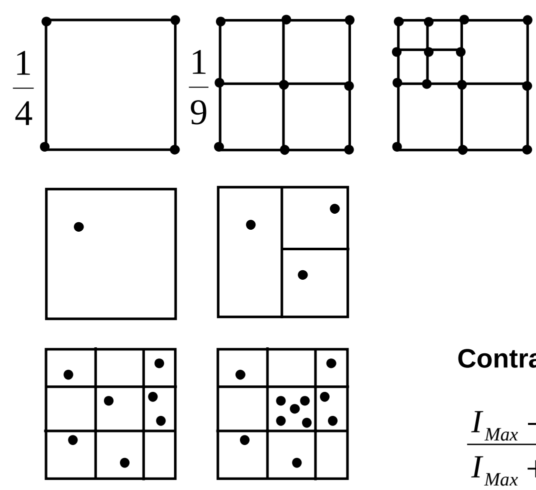
Recurse

Kajiya

Split node

Mitchell

Shotgun blast



Contrast

$$\frac{I_{Max} - I_{Min}}{I_{Max} + I_{Min}}$$

Other criteria …