Overview

Last Lecture

- **Statistical sampling and Monte Carlo integration**

Today

- **Variance reduction**
- **Importance sampling**
- **Discrepancy and Quasi-Monte Carlo**
- **Multidimensional sampling patterns**

Latter

- **Path tracing for interreflection**
- **Density estimation**

Variance Reduction

Efficiency measure

$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

Some techniques

- **Estimators**
- **Expected values vs. rejection sampling**
- **Importance sampling**
- **Sampling patterns: stratified, correlated, antithetic**

Biasing

Biasing the sampling process

$$X_i \sim p(x) \qquad Y_i = \frac{f(X_i)}{p(X_i)}$$

$$\begin{aligned} E[Y_i] &= E\left[\frac{f(X_i)}{p(X_i)}\right] \\ &= \int \left[\frac{f(X_i)}{p(X_i)}\right] p(x)\,dx \\ &= \int f(x)\,dx \\ &= I \end{aligned}$$

Importance Sampling

Variance

$$V[f] = E[f^2] - E^2[f]$$

$$E[Y_i^2] = \int \left[\frac{f(X_i)}{p(X_i)}\right]^2 p(x)\,dx$$

Zero variance biasing

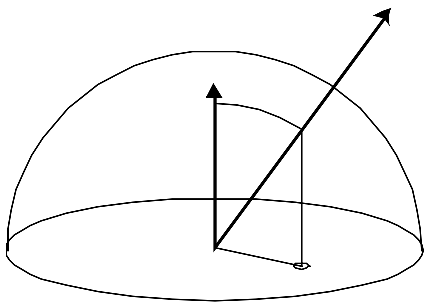
$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$V[\tilde{f}^2] = 0$$

$$\begin{aligned} E[\tilde{f}^2] &= \int \left[\frac{f(x)}{\tilde{p}(x)}\right]^2 \tilde{p}(x)\,dx \\ &= \int \left[\frac{f(x)}{f(x)/E[f]}\right]^2 \frac{f(x)}{E[f]} dx \\ &= E^2[f] \end{aligned}$$

Irradiance

Generate cosine weighted distribution

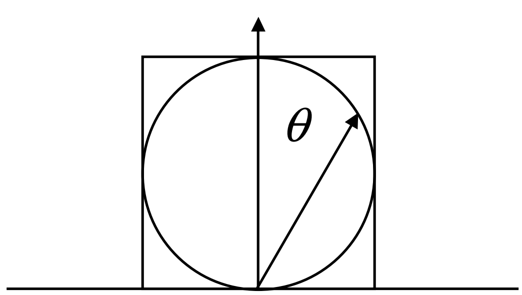


$$p(\omega)\,d\omega = \cos\theta\,d\omega$$

$$E = \int_{H^2} L_i(\omega_i)\cos\theta_i\,d\omega_i$$

Irradiance

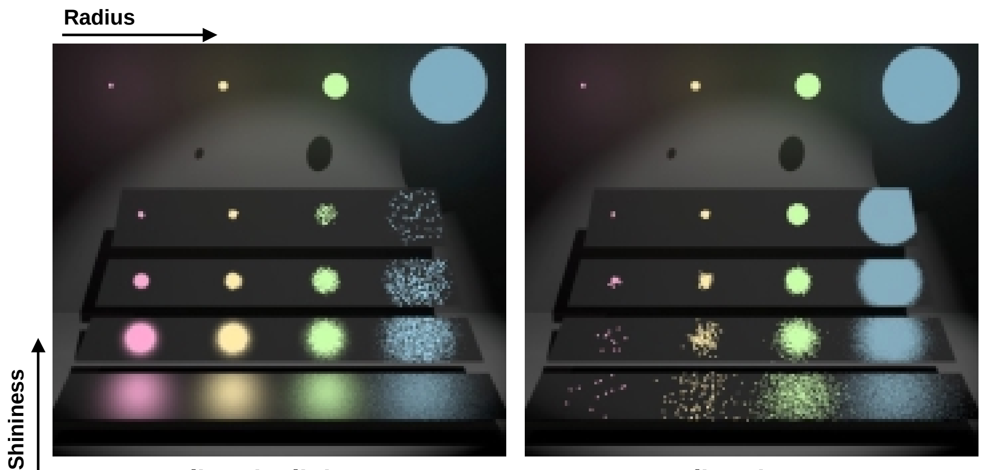
Generate cosine weighted distribution



$\cos\theta$

Multiple Importance Sampling

Reflection of a circular light source by a rough surface



Sampling the light source **Sampling the BRDF**

$$\int f(x)g(x)\,dx$$

Multiple Importance Sampling

Two sampling techniques

$$X_{1,i} \sim p_1(x) \qquad X_{2,i} \sim p_2(x)$$

$$Y_{1,i} = \frac{f(X_{1,i})}{p_1(X_{1,i})} \qquad Y_{2,i} = \frac{f(X_{2,i})}{p_2(X_{2,i})}$$

Form weighted combination of samples

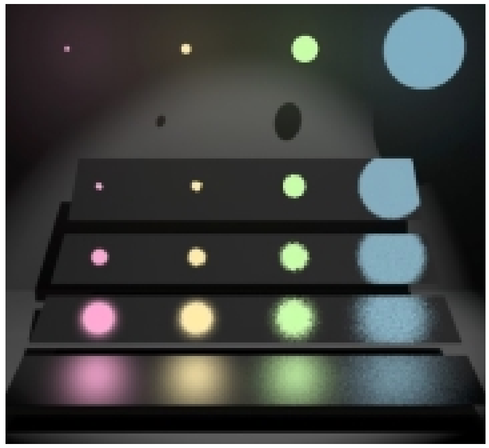
$$Y_i = w_1 Y_{1,i} + w_2 Y_{2,i}$$

The balance heuristic

$$w_i(x) = \frac{p_i(x)}{p_1(x) + p_2(x)} \Rightarrow p(x) = w_1(x)p_1(x) + w_2(x)p_2(x)$$

Multiple Importance Sampling

Combine both sampling methods



Source: Veach and Guibas

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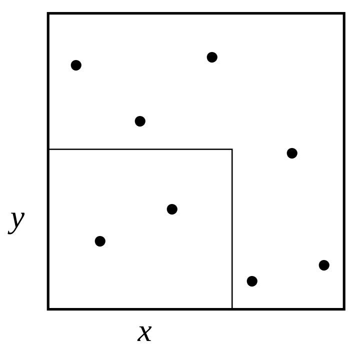
Discrepancy



$$\Delta(x,y) = \frac{n(x,y)}{N} - xy$$

$A = xy$

$n(x,y)$ number of samples in A

$$D_N = \max_{x,y} \left| \Delta(x,y) \right|$$

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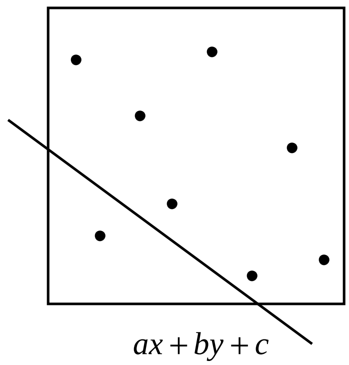
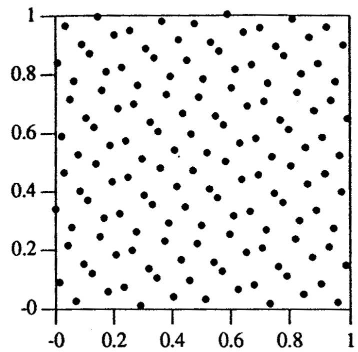
Edge Discrepancy





Zaremba-Hammersley Points

Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer *i* in integer base *b*

$$i = d_i \cdots d_2 d_1 d_0$$
$$\phi_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i$$

			$\phi_2(i)$
1	1	.1	1/2
2	10	.01	1/4
3	11	.11	3/4
4	100	.001	3/8
5	101	.101	5/8

Hammersley points

$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \cdots)$$

$$D_N = O\left(\frac{\log^{d-1} N}{N}\right)$$

Halton points (sequential)

$$(\phi_2(i), \phi_3(i), \phi_5(i), \cdots)$$

$$D_N = O\left(\frac{\log^{d} N}{N}\right)$$

Low-Discrepancy Patterns

Process	16 points	256 points	1600 points
Zaremba	0.0504	0.00478	0.00111
Jittered	0.0538	0.00595	0.00146
Poisson-Disk	0.0613	0.00767	0.00241
N-Rooks	0.0637	0.0123	0.00488
Random	0.0924	0.0224	0.00866

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended
Story still unclear in higher dimensions?

Theorem on Total Variation

Theorem:

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \int f(x)\,dx \right| \le V(f) D_N$$

Proof: Integrate by parts

$$\int f(x) \ \left[\frac{\delta(x - x_i)}{N} - 1\right] dx \qquad \frac{\partial \Delta(x)}{\partial x} = \frac{\delta(x - x_i)}{N} - 1$$

$$= \int f(x) \frac{\partial \Delta(x)}{\partial x} dx$$

$$= f\Delta\Big|_0^1 - \int \frac{\partial f(x)}{\partial x} \Delta(x) \cdots dx = -\int \frac{\partial f(x)}{\partial x} \Delta(x)\,dx$$

$$\le D_N \int \left| \frac{\partial f(x)}{\partial x} \right| dx = V(f) D_N$$

Block Design

a	*d*	*c*	*b*
b	*a*	*d*	*c*
c	*b*	*a*	*d*
d	*c*	*b*	*a*

Latin Square

Alphabet of size *n*

Each symbol appears exactly once in each row and column

Improves discrepancy

a			
		a	
	a		
			a

N-Rook Pattern

$(\pi_x = \{1,2,3,4\}, \pi_y = \{4,2,3,1\})$

Incomplete block design

Replaced N^d samples with N samples

Permutations: $(\pi_1(i), \pi_2(i), \cdots \pi_d(i))$

Generalizations: N-queens, 2D projection

Space-time Patterns

6	0	2	3
3	4	2	8
5	0	7	1
5	9	4	

Cook Pattern

5	8	5	2
4	3	4	9
0	3	0	7
	6	1	2

Pan-diagonal Magic Square

Fully populate *(x,y)* samples

- **Recall blue noise good**
  - Perceptually pleasing
  - Filtered during resampling
- **Jitter to achieve blue noise**

Distribute *t* samples

- **Decorrelate space and time**
- **Nearby samples in space should differ greatly in time**

Mitchell (1991) designs

Views of Integration

1. Signal processing
   - Sampling and reconstruction, aliasing and antialiasing
   - Blue noise good
2. Statistical sampling
   - Monte Carlo: variance, central limit theorem
   - Adaptive sampling criteria, $N^{-1/2}$, high dimensional sampling
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Numerical
   - Quadrature/Integration rules
   - Smooth functions