

Overview

Last Lecture

- Statistical sampling and Monte Carlo integration

Today

- Variance reduction
- Importance sampling
- Discrepancy and Quasi-Monte Carlo
- Multidimensional sampling patterns

Latter

- Path tracing for interreflection
- Density estimation

Variance Reduction

Efficiency measure

$$\textit{Efficiency} \propto \frac{1}{\textit{Variance} \bullet \textit{Cost}}$$

Some techniques

- Estimators
- Expected values vs. rejection sampling
- Importance sampling
- Sampling patterns: stratified, correlated, antithetic

Biasing

Biasing the sampling process

$$X_i \sim p(x) \quad Y_i = \frac{f(X_i)}{p(X_i)}$$

$$\begin{aligned} E[Y_i] &= E\left[\frac{f(X_i)}{p(X_i)}\right] \\ &= \int \left[\frac{f(X_i)}{p(X_i)}\right] p(x) dx \\ &= \int f(x) dx \\ &= I \end{aligned}$$

Importance Sampling

Variance

$$V[f] = E[f^2] - E^2[f]$$

$$E[Y_i^2] = \int \left[\frac{f(X_i)}{p(X_i)} \right]^2 p(x) dx$$

Zero variance biasing

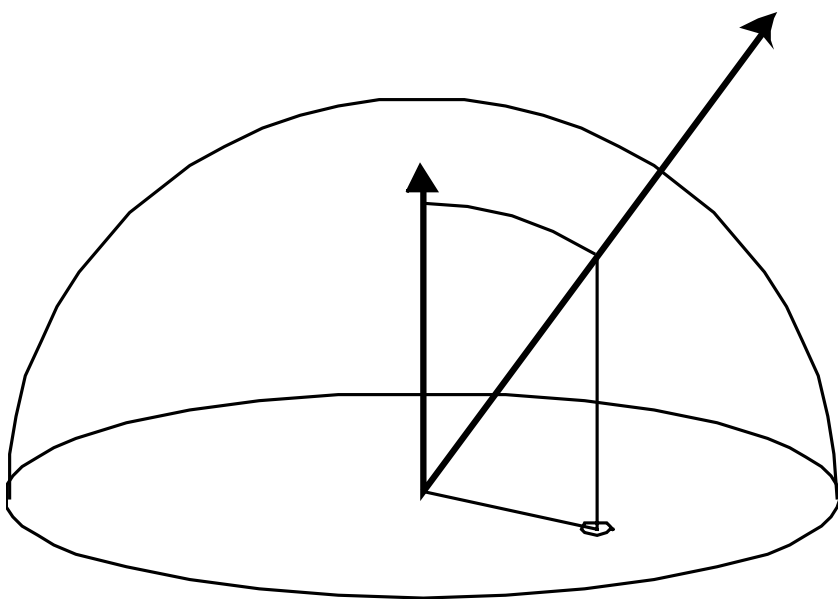
$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$V[\tilde{f}^2] = 0$$

$$\begin{aligned} E[\tilde{f}^2] &= \int \left[\frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x) dx \\ &= \int \left[\frac{f(x)}{f(x)/E[f]} \right]^2 \frac{f(x)}{E[f]} dx \\ &= E^2[f] \end{aligned}$$

Irradiance

Generate cosine weighted distribution

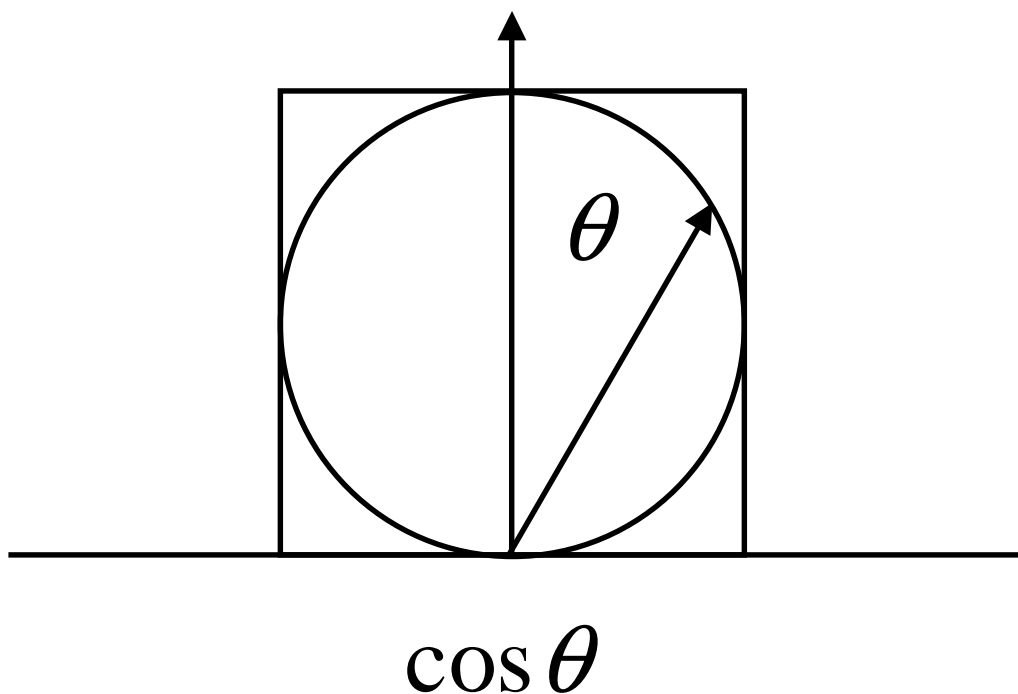


$$p(\omega) d\omega = \cos \theta d\omega$$

$$E = \int_{H^2} L_i(\omega_i) \cos \theta_i d\omega_i$$

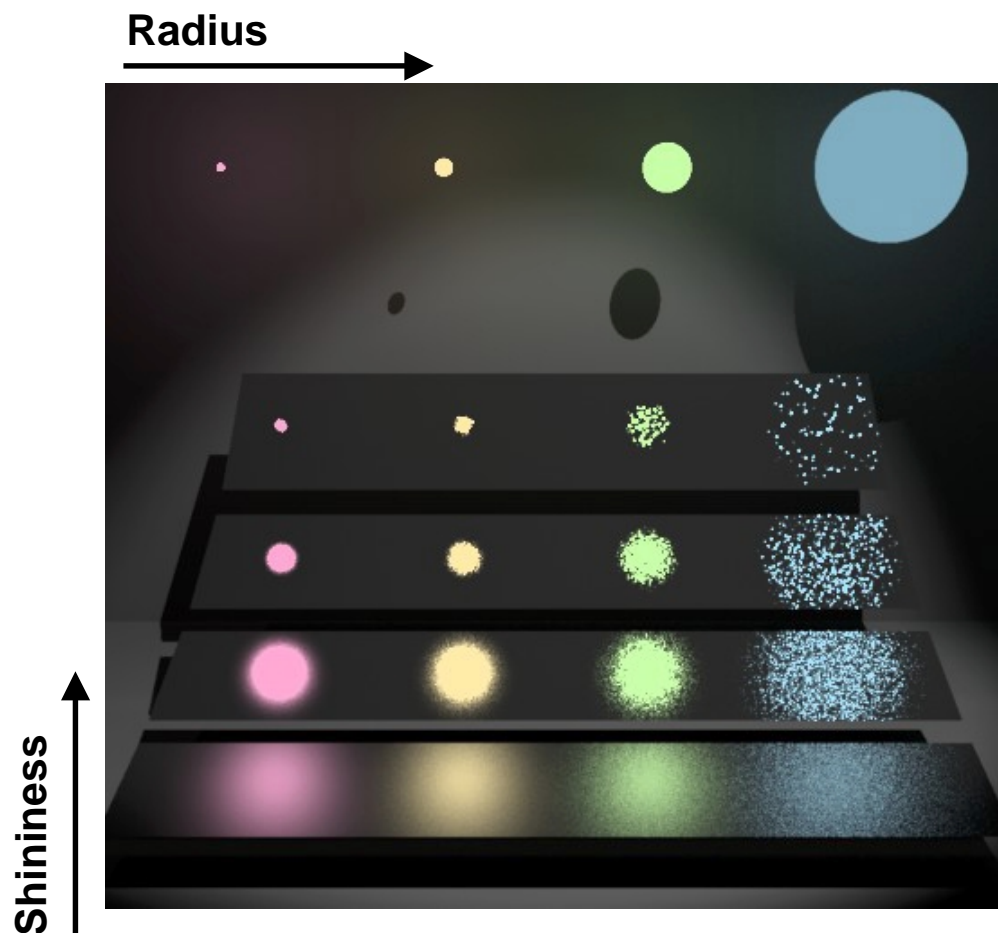
Irradiance

Generate cosine weighted distribution

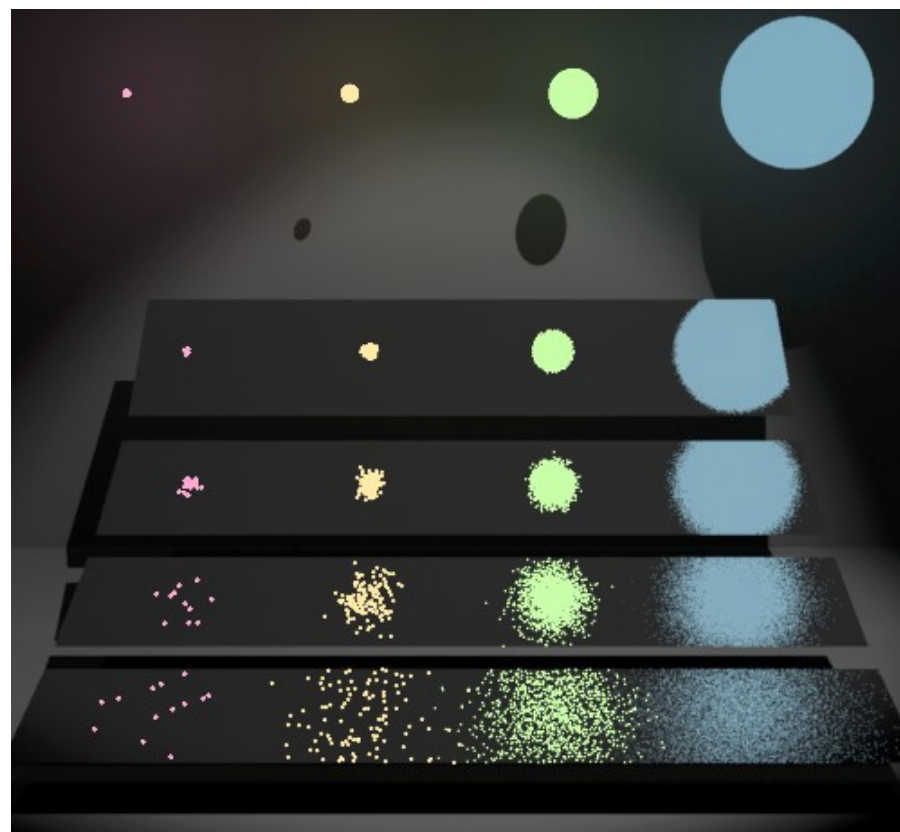


Multiple Importance Sampling

Reflection of a circular light source by a rough surface



Sampling the light source



Sampling the BRDF

$$\int f(x)g(x)dx$$

Multiple Importance Sampling

Two sampling techniques

$$\begin{aligned} X_{1,i} &\sim p_1(x) & X_{2,i} &\sim p_2(x) \\ Y_{1,i} &= \frac{f(X_{1,i})}{p_1(X_{1,i})} & Y_{2,i} &= \frac{f(X_{2,i})}{p_2(X_{2,i})} \end{aligned}$$

Form weighted combination of samples

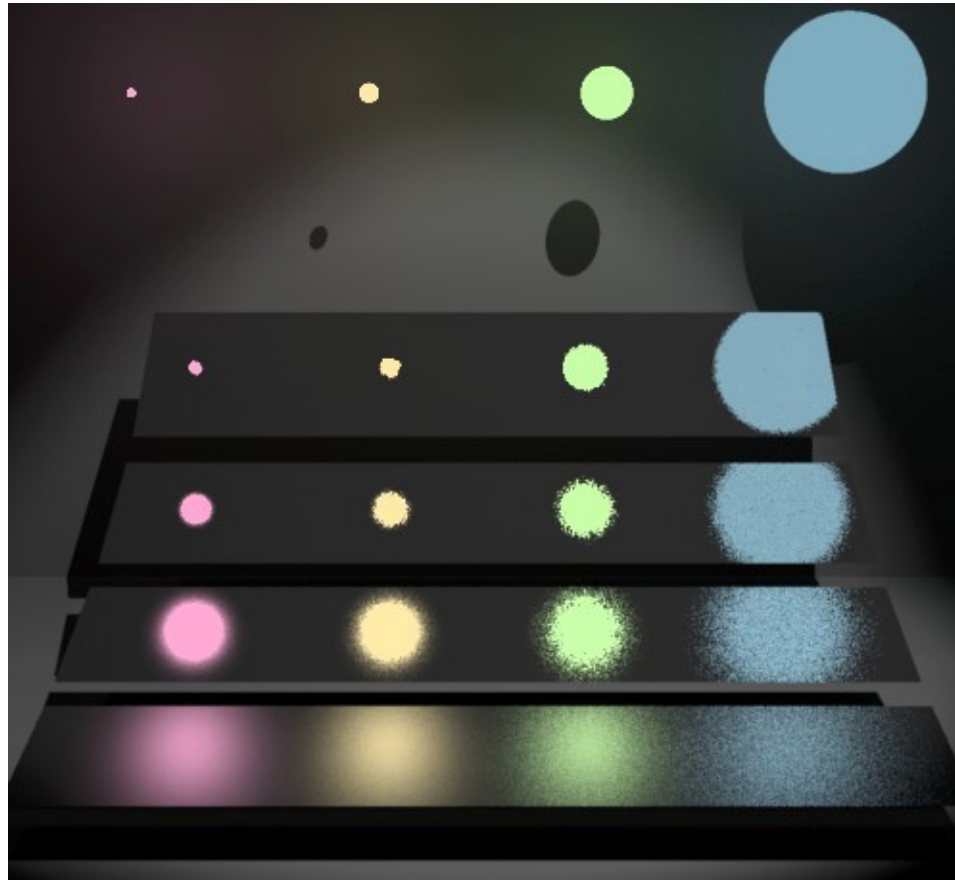
$$Y_i = w_1 Y_{1,i} + w_2 Y_{2,i}$$

The balance heuristic

$$w_i(x) = \frac{p_i(x)}{p_1(x) + p_2(x)} \Rightarrow p(x) = w_1(x) p_1(x) + w_2(x) p_2(x)$$

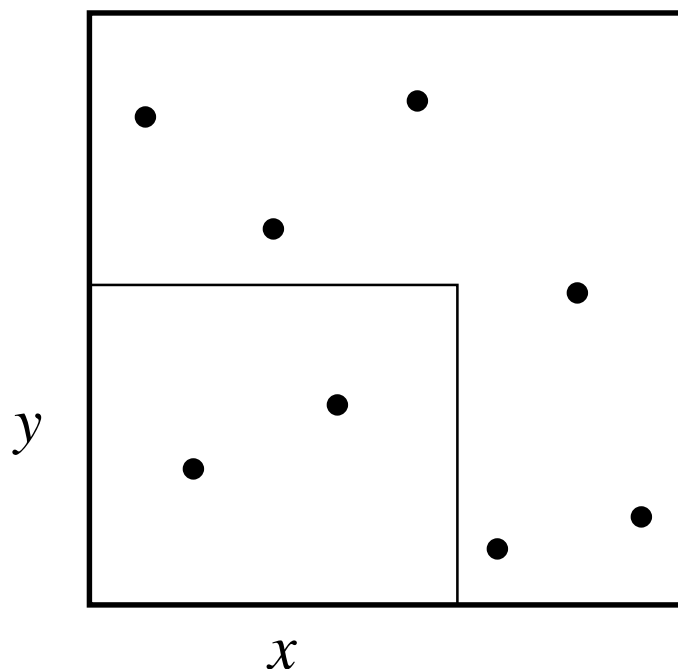
Multiple Importance Sampling

Combine both sampling methods



Source: Veach and Guibas

Discrepancy



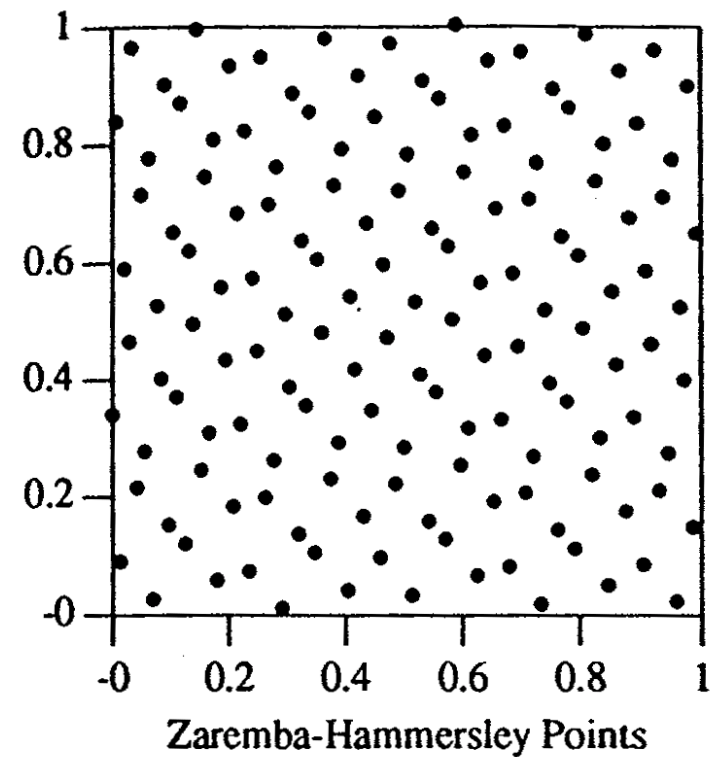
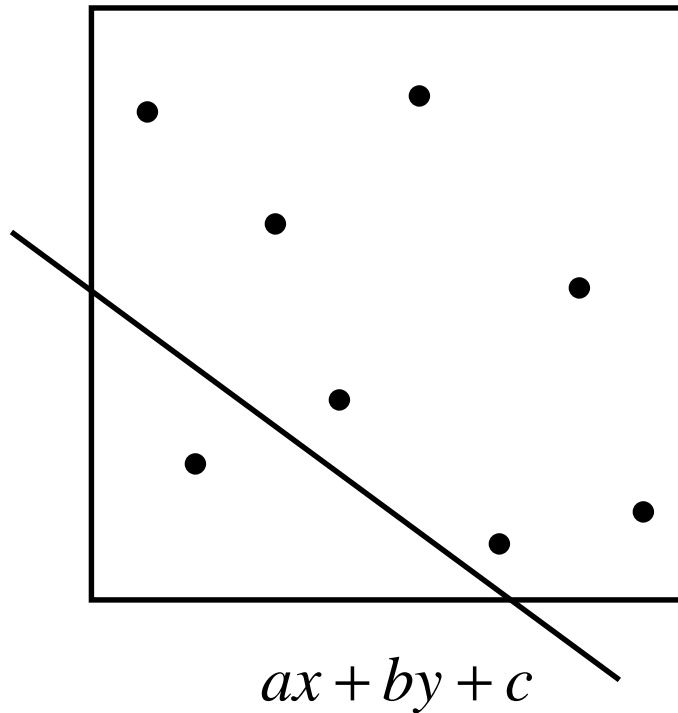
$$\Delta(x, y) = \frac{n(x, y)}{N} - xy$$

$$A = xy$$

$n(x, y)$ number of samples in A

$$D_N = \max_{x, y} |\Delta(x, y)|$$

Edge Discrepancy



**Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples**

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer i in integer base b

$$i = d_i \cdots d_2 d_1 d_0$$

$$\phi_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i$$

$$\phi_2(i)$$

1	1	.1	1/2
2	10	.01	1/4
3	11	.11	3/4
4	100	.001	3/8
5	101	.101	5/8

Hammersley points

$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \dots)$$

Halton points (sequential)

$$(\phi_2(i), \phi_3(i), \phi_5(i), \dots)$$

$$D_N = O\left(\frac{\log^{d-1} N}{N}\right)$$

$$D_N = O\left(\frac{\log^d N}{N}\right)$$

Low-Discrepancy Patterns

Process	16 points	256 points	1600 points
Zaremba	0.0504	0.00478	0.00111
Jittered	0.0538	0.00595	0.00146
Poisson-Disk	0.0613	0.00767	0.00241
N-Rooks	0.0637	0.0123	0.00488
Random	0.0924	0.0224	0.00866

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$

Zaremba converges faster and has lower discrepancy

Zaremba has a relatively poor blue noise spectra

Jittered and Poisson-Disk recommended

Story still unclear in higher dimensions?

Theorem on Total Variation

Theorem:

$$\left| \frac{1}{N} \sum_{i=1}^N f(X_i) - \int f(x) dx \right| \leq V(f) D_N$$

Proof: Integrate by parts

$$\begin{aligned} & \int f(x) \left[\frac{\delta(x - x_i)}{N} - 1 \right] dx & \frac{\partial \Delta(x)}{\partial x} &= \frac{\delta(x - x_i)}{N} - 1 \\ &= \int f(x) \frac{\partial \Delta(x)}{\partial x} dx \\ &= f \Delta \Big|_0^1 - \int \frac{\partial f(x)}{\partial x} \Delta(x) \cdots dx = - \int \frac{\partial f(x)}{\partial x} \Delta(x) dx \\ &\leq D_N \int \left| \frac{\partial f(x)}{\partial x} \right| dx = V(f) D_N \end{aligned}$$

Block Design

a	d	c	b
b	a	d	c
c	b	a	d
d	c	b	a

Latin Square

Alphabet of size n

Each symbol appears exactly once in each row and column

Improves discrepancy

a			
		a	
	a		
			a

N-Rook Pattern

Incomplete block design

Replaced N^d samples with N samples

Permutations: $(\pi_1(i), \pi_2(i), \dots, \pi_d(i))$

Generalizations: N-queens, 2D projection

$$(\pi_x = \{1, 2, 3, 4\}, \pi_y = \{4, 2, 3, 1\})$$

Space-time Patterns

6	0	2	3
3	4	2	8
5	0	7	1
5	9	4	

Cook Pattern

5	8	5	2
4	3	4	9
0	3	0	7
	6	1	2

Pan-diagonal Magic Square

Fully populate (x,y) samples

- **Recall blue noise good**
Perceptually pleasing
Filtered during resampling
- **Jitter to achieve blue noise**

Distribute t samples

- **Decorrelate space and time**
- **Nearby samples in space
should differ greatly in time**

Mitchell (1991) designs

Views of Integration

1. Signal processing

- Sampling and reconstruction, aliasing and antialiasing
- Blue noise good

2. Statistical sampling

- Monte Carlo: variance, central limit theorem
- Adaptive sampling criteria, $N^{-1/2}$, high dimensional sampling

3. Quasi Monte Carlo

- Discrepancy
- Asymptotic efficiency in high dimensions

4. Numerical

- Quadrature/Integration rules
- Smooth functions