Monte Carlo I: Foundations

cs348b
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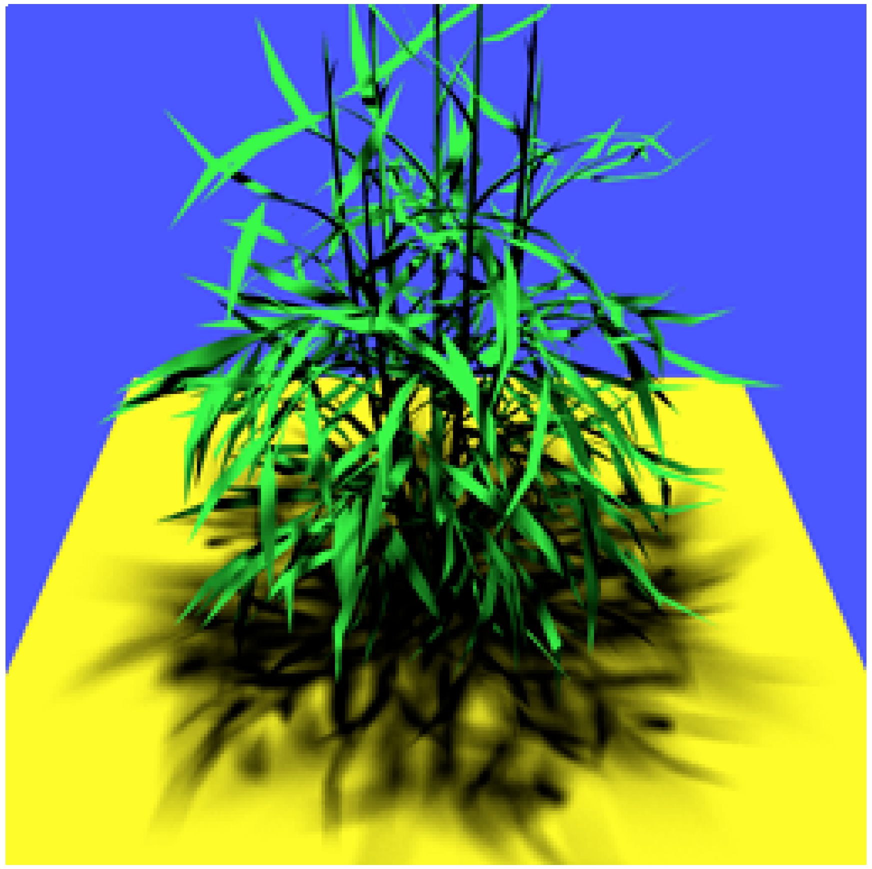
Integration Challenges in Rendering

- Complex light geometry
- Complex visibility
- Discontinuous integrands
- Many times per pixel

$$E(x) = \int_{H^2} L_i(x, \omega) |\cos\theta| d\omega$$



Advantages of MC Integration

- Few restrictions on the integrand
  - Doesn't need to be continuous, smooth, ...
  - Only need to be able to evaluate at a point
- Extends to high-dimensional problems
- Conceptually straightforward
- Efficient for solving at just a few points

Disadvantages of MC

- Noisy
- Slow convergence
- Good implementation is hard
  - Debugging
  - Choosing appropriate techniques

Overview

- Random variables
  - Sampling from distributions
- Basic MC estimator
- Variance
  - Definition and causes
  - Variance reduction: importance sampling
- Efficiency
  - Splitting & Russian Roulette

Random Variables

- Definition
  - $\bar{x}$ is chosen by some random process
  - $\bar{x} \sim p(x)$ probability distribution function (pdf)
  - pdf is non-negative, integrates to one
  - *not* necessarily <= 1!
- Pdf examples
  - $p(x) = 1, 0 \leq x < 1$
  - $p(x) = 1/2, 5 \leq x < 7$
  - $p(x) = \cos(x), 0 \leq x < \pi/2$
  - $p(x) = \delta(x)$

Probability Distributions

- Discrete
  - Events $\bar{x}_i$ with probability $p_i \geq 0$ $\quad \sum_{i=1}^{N} p_i = 1$
  - Cumulative distribution $P_j = \sum_{i=1}^{j} p_i$
  - Can choose event with uniform random num. ξ
- Continuous

$$\bar{x} \sim p(x) \qquad p(x) \geq 0$$

$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(\bar{x} < x) \qquad \Pr(\alpha \leq \bar{x} \leq \beta) = P(\beta) - P(\alpha)$$

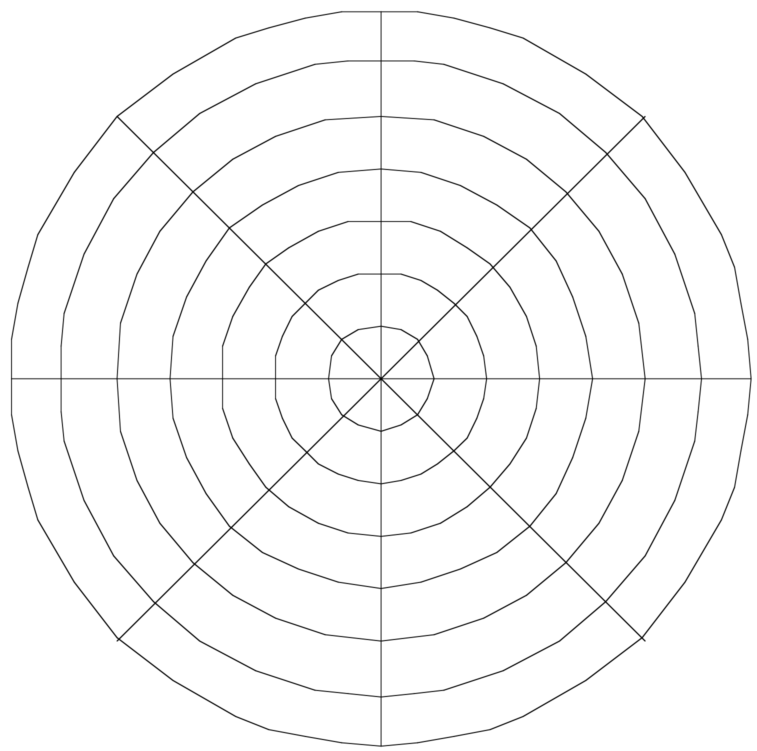
Sampling the Power Function

- Given $f(x) = x^n$
- Normalize to find pdf p(x)
- Find cumulative distribution function P(x)
- Solve to find $\bar{x} = P^{-1}(\xi)$

- Trick: $\bar{x} = \max_n(\xi_1, \ldots, \xi_n)$

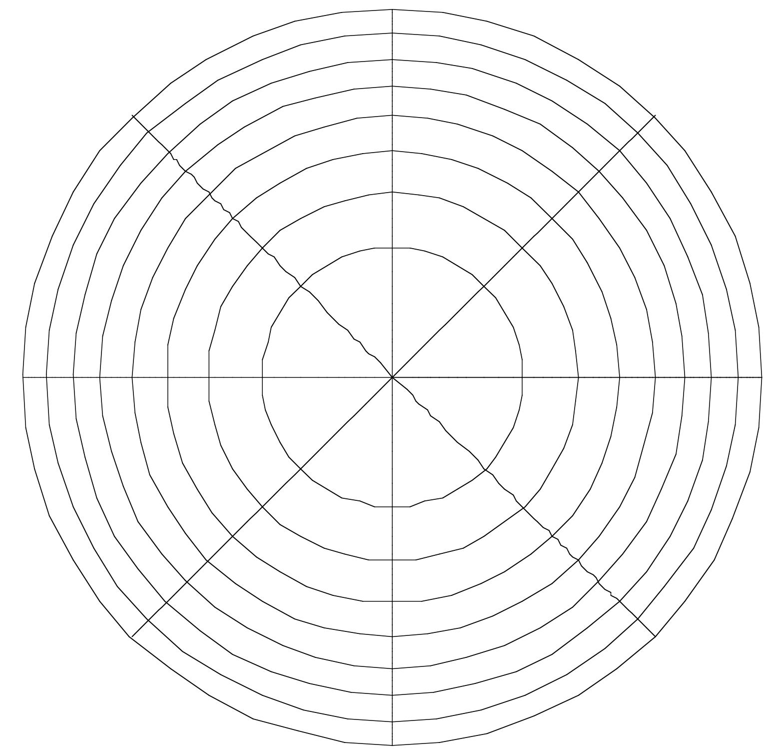
Sampling a Circle

Wrong



$$\theta = 2\pi\xi_1$$
$$r = \xi_2$$

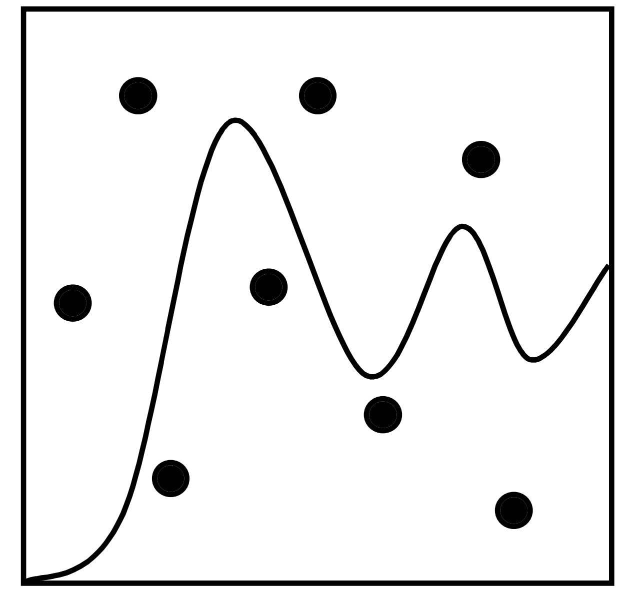
Right



$$\theta = 2\pi\xi_1$$
$$r = \sqrt{\xi_2}$$

Basic Rejection Sampling

- Pick ξ_1 and ξ_2
- Accept ξ_1 if $\xi_2 < c \cdot f(\xi_1)$



- Applications
  - Sampling points in a circle
  - Sampling directions on the sphere

Expected Value

- Definition

$$E[f] \equiv \int_0^1 f(x)p(x)dx$$

- Key property: linearity
- Unbiased MC Estimator:

$$E\left[\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right] = \int_0^1 f(x)dx$$

Unbiased MC Estimator

$$E\left[\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right] = \frac{1}{N}\sum_{i=1}^{N} E[f(x_i)] \qquad x_i \sim p(x)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)p(x)dx$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)dx$$ if p(x) is uniform

$$= \int_0^1 f(x)dx$$

Natural extension to multiple dimensions...

Direct Lighting: Area Sampling

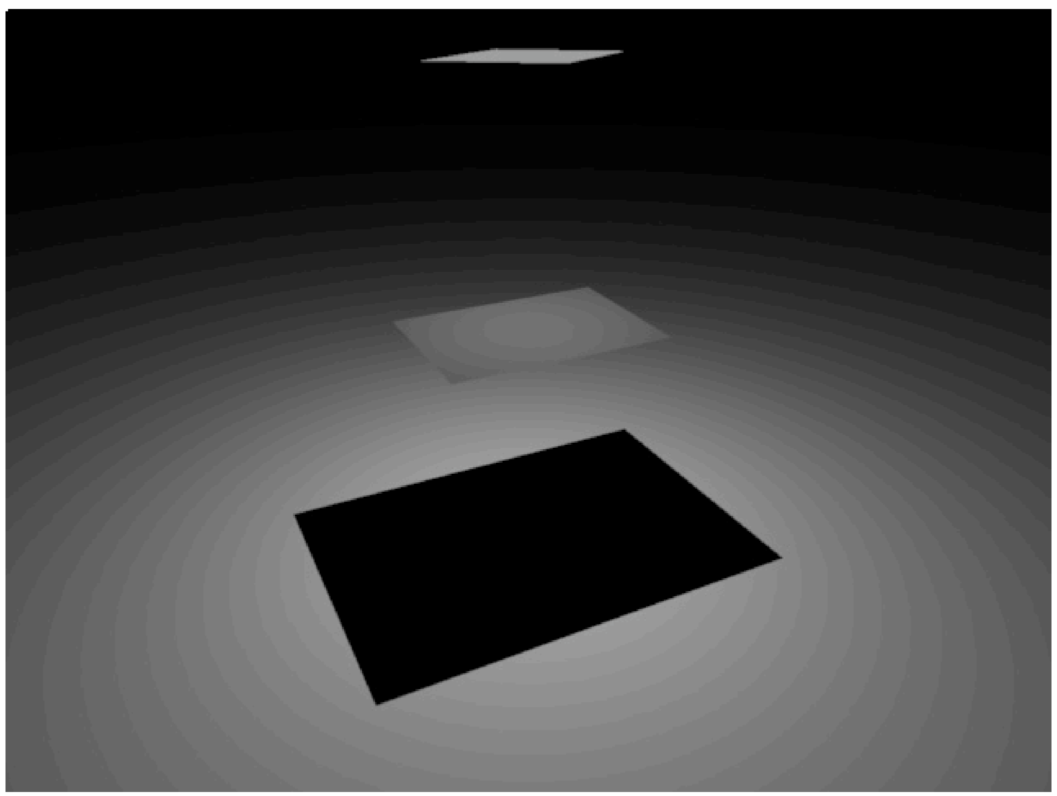
$$E(x) = \int_\Omega L(x,\omega)\cos\theta d\omega$$
$$= \int_A L(x_o,\omega_o)\frac{\cos\theta\cos\theta_o}{|x-x_o|^2}V(x,x_o)dA(x_o)$$

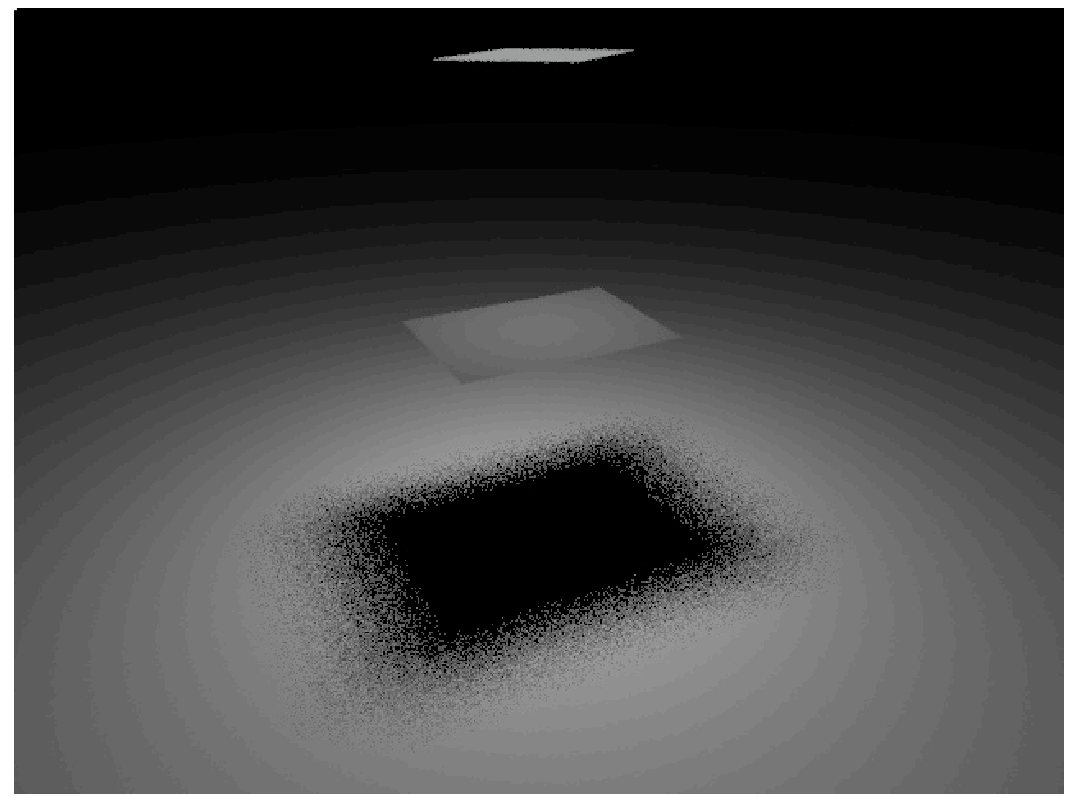
Sample light uniformly by area

$$E(x) \approx \frac{1}{N}\sum_{i=1}^{N} L(x_{o[i]},\omega_{o[i]})\frac{\cos\theta\cos\theta_{o[i]}}{|x-x_{o[i]}|^2}V(x,x_{o[i]})$$

Direct Lighting

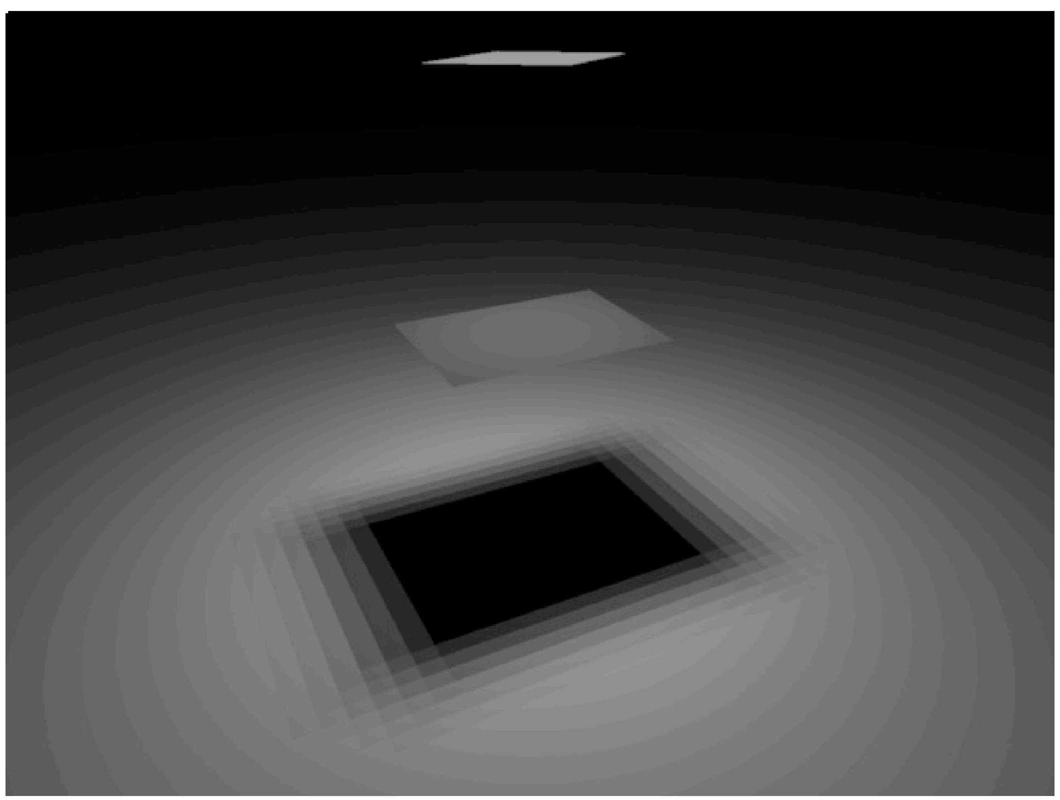
4 eye rays, 1 shadow ray

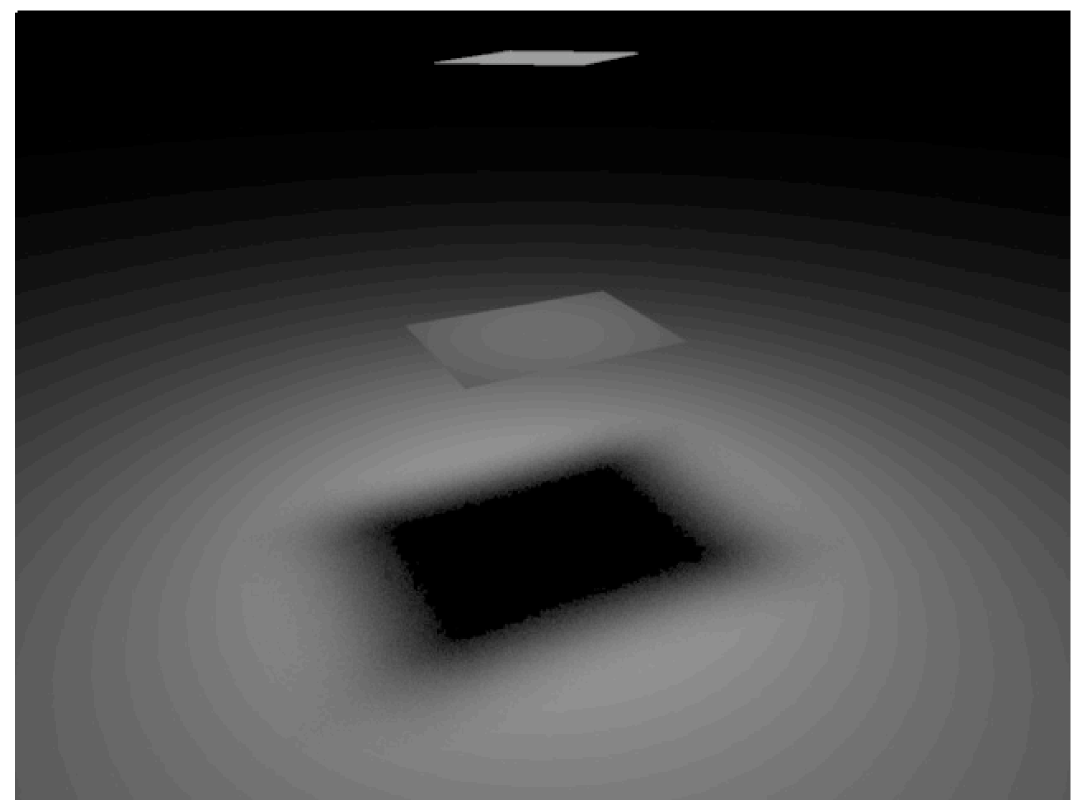




Direct Lighting

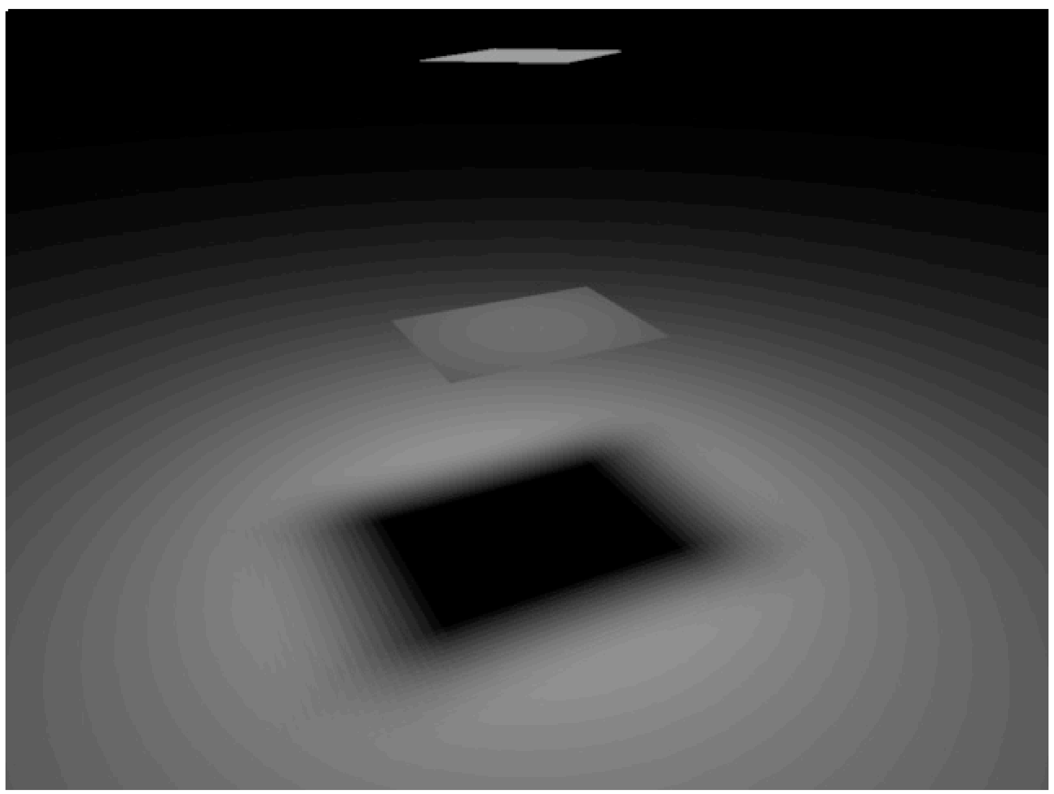
4 eye rays, 16 shadow rays

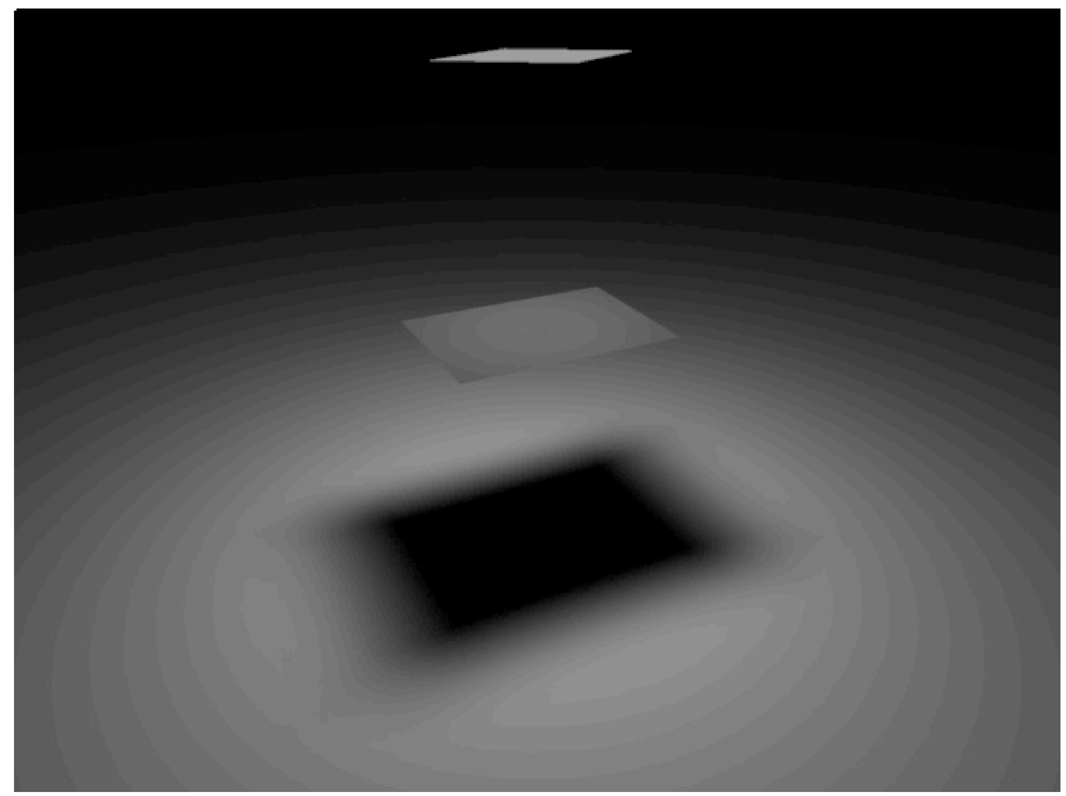




Direct Lighting

4 eye rays, 64 shadow rays





Variance

- Definition

$$V[Y] \equiv E[(Y - E[Y])^2] = E[Y^2] - E[Y]^2$$

- Properties

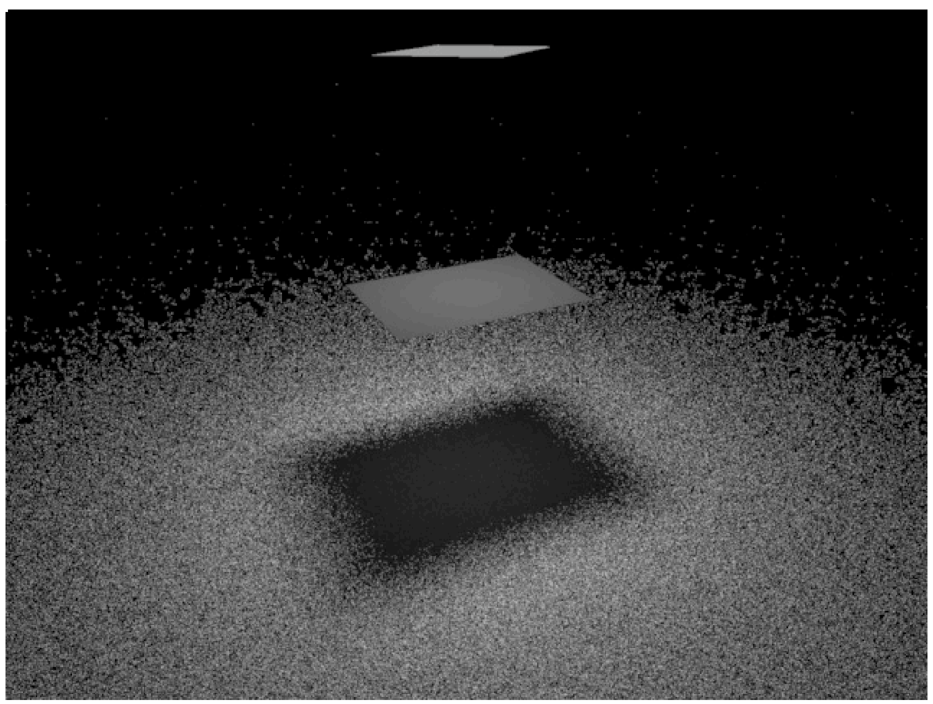
$$V[\sum Y_i] = \sum V[Y_i] \qquad V[aY] = a^2 \sum V[Y]$$

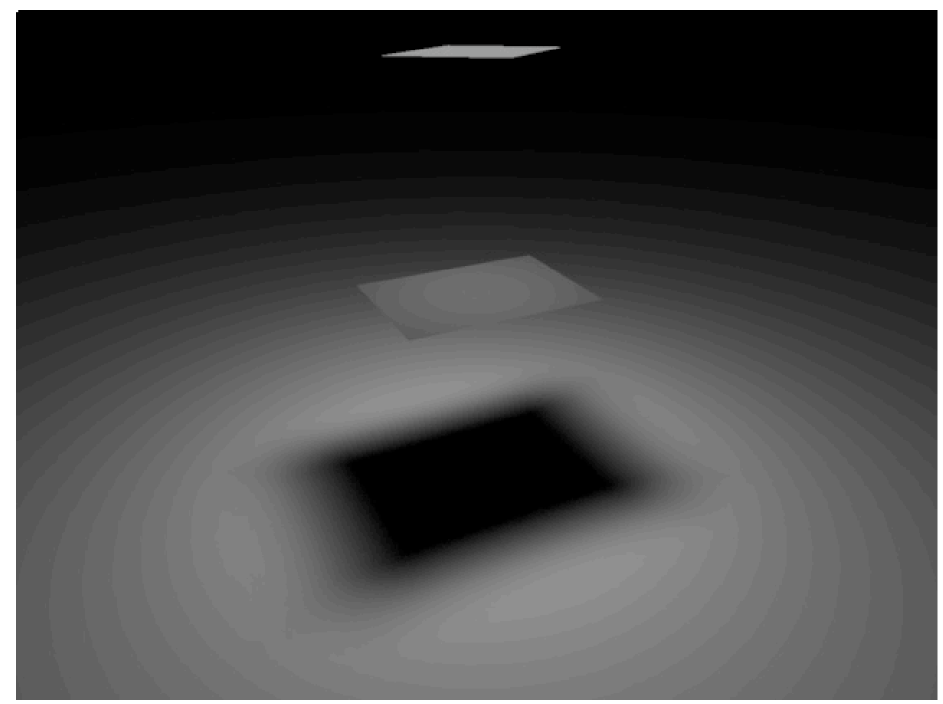
- Variance decreases with sample size

$$V\left[\frac{1}{N}\sum Y_i\right] = \frac{1}{N^2}\sum V[Y_i] = \frac{1}{N}V[Y]$$

Direct Lighting

- Directional sampling wastes samples where the integrand is zero





Basic MC strategy: focus samples where integrand is high (easier said than done)

Improving Efficiency: Splitting

- Name from neutron transport applications
- Take additional samples in some dimensions

$$\int_0^1 \int_0^1 f(x)g(y)dxdy \approx \frac{1}{N} \sum f(x_i)g(y_i)$$

$$\approx \frac{1}{N_f} \sum_{i=1}^{N_f} f(x_i) \left(\frac{1}{N_g} \sum_{j=1}^{N_g} g(y_j) \right)$$

- e.g. multiple shadow rays

Improving Efficiency: Russian Roulette

- Randomly skip computations with low expected value...in an unbiased manner
- Algorithm:
  - Determine probability of skipping p_s
  - Choose uniform random number ξ
  - Skip if $\xi < p_s$
  - Otherwise estimate is $\dfrac{f(x_i)}{1-p_s}$
- Expected value

$$p_s \cdot 0 + (1-p_s) \cdot E[f/(1-p_s)] = E[f]$$

Importance Sampling

- Try to sample where integrand is large

$$E\left[\frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}\right] = \frac{1}{N}\sum_{i=1}^{N}E\left[\frac{f(x_i)}{p(x_i)}\right] \qquad x_i \sim p(x)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 \frac{f(x)}{p(x)}p(x)dx$$

$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)dx$$

$$= \int_0^1 f(x)dx$$