The Rendering Equation

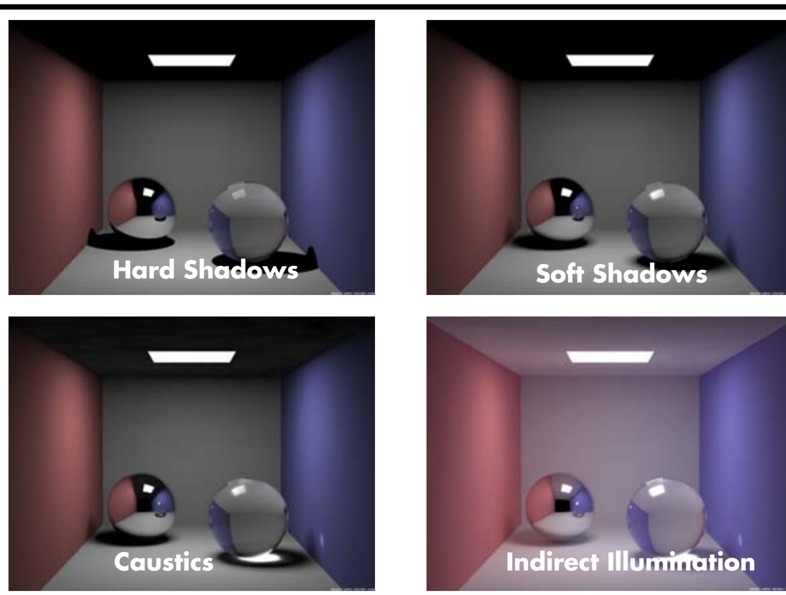
Direct (*local*) illumination

- **Light directly from light sources**
- **No shadows**

Indirect (*global*) illumination

- **Hard and soft shadows**
- **Diffuse interreflections (radiosity)**
- **Glossy interreflections (caustics)**

Lighting Effects



Early Radiosity

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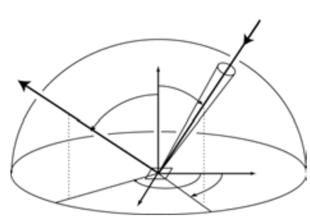
Challenge

To evaluate the reflection equation
the incoming radiance must be known

$$L_r(x,\omega_r) = \int_{H^2} f_r(x,\omega_i \rightarrow \omega_r) L_i(x,\omega_i) \cos\theta_i \, d\omega_i$$



To evaluate the incoming radiance
the reflected radiance must be known

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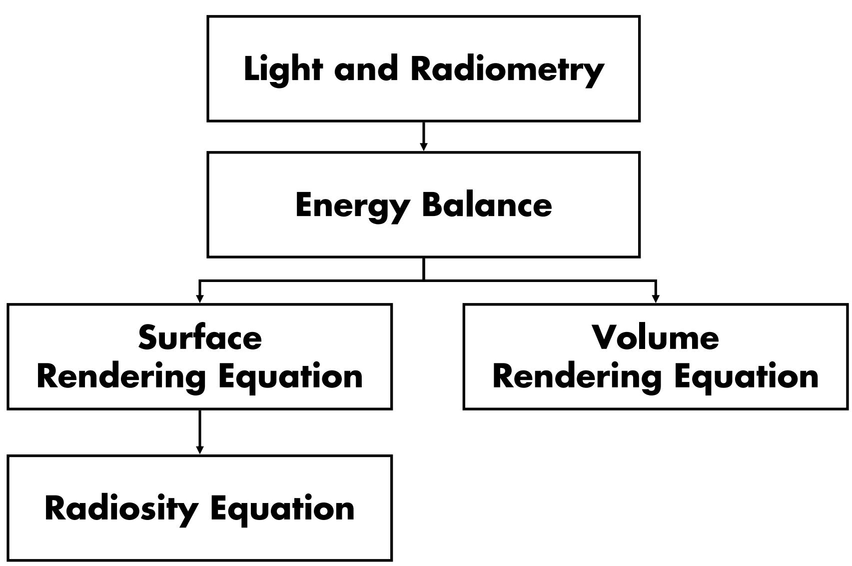
To The Rendering Equation

Questions

1. How is light measured?
2. How is the spatial distribution of light energy described?
3. How is reflection from a surface characterized?
4. What are the conditions for equilibrium flow of light in an environment?

The Grand Scheme



Balance Equation

Accountability

[***outgoing***] - [***incoming***] = [***emitted***] - [***absorbed***]

- **Macro level**

The total light energy put into the system must equal the energy leaving the system (usually, via heat).

$$\Phi_o - \Phi_i = \Phi_e - \Phi_a$$

- **Micro level**

The energy flowing into a small region of phase space must equal the energy flowing out.

$$B(x) - E(x) = B_e(x) - E_a(x)$$

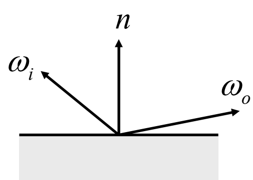
$$L_o(x,\omega) - L_i(x,\omega) = L_e(x,\omega) - L_a(x,\omega)$$

Surface Balance Equation

[***outgoing***] = [***emitted***] + [***reflected***]

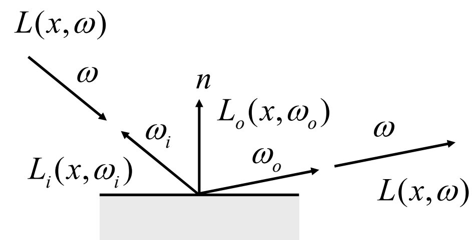
$$\begin{aligned} L_o(x,\omega_o) &= L_e(x,\omega_o) + L_r(x,\omega_o) \\ &= L_e(x,\omega_o) + \int_{H^2} f_r(x,\omega_i \to \omega_o) L_i(x,\omega_i) \cos\theta_i \, d\omega_i \end{aligned}$$

Direction Conventions



$f_r(\omega_i \to \omega_o)$

BRDF



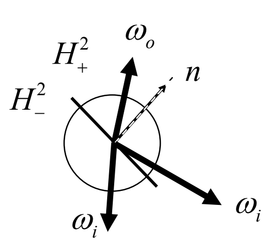
$$L_i(x,\omega_i) = L(x,-\omega_i)$$
$$L(x,\omega) = L_o(x,\omega_o)$$

Surface vs. Field Radiance

Surface Balance Equation

[*outgoing*] = [*emitted*] + [*reflected*] + [*transmitted*]

$$L_o(x,\omega_o) = L_e(x,\omega_o) + L_r(x,\omega_o) + L_t(x,\omega_o)$$



$$L_r(x,\omega_o) = \int_{H_+^2} f_r(x,\omega_i \to \omega_o) L_i(x,\omega_i) \cos\theta_i \, d\omega_i$$

$$L_t(x,\omega_o) = \int_{H_-^2} f_t(x,\omega_t \to \omega_o) L_i(x,\omega_t) \cos\theta_t \, d\omega_t$$

BTDF

$H_+^2(n) \quad \omega_i \bullet n(x) > 0$

$H_-^2(n) \quad \omega_i \bullet n(x) < 0$

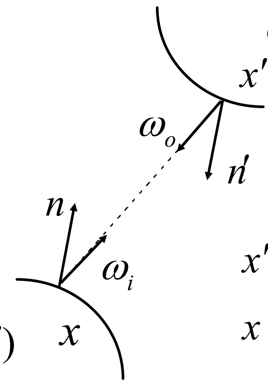
Two-Point Geometry

$$\omega(x, x') = \omega(x \to x') = \frac{x' - x}{|x' - x|}$$

Ray Tracing

$$x^*(x, \omega)$$

$$\omega_o = \omega(x', x)$$



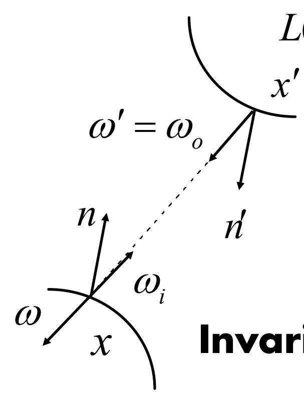
$$x' = x^*(x, \omega_i)$$

$$x = x^*(x', \omega_o)$$

$$\omega_i = \omega(x, x')$$

Coupling Equations

$$L(x', \omega') = L_o(x', \omega_o)$$



Invariance of radiance

$$L_i(x, \omega_i) = L(x, -\omega_i)$$

$$L(x, \omega) = L(x', \omega')$$

The Rendering Equation

Directional form

$$L(x,\omega) = L_e(x,\omega) + \int_{H^2} f_r(x,\omega' \to \omega)\, L(x^*(x,\omega'),-\omega') \cos\theta' \, d\omega'$$

Integrate over hemisphere of directions

Transport operator i.e. ray tracing

The Rendering Equation

Surface form

$$L(x',x) = L_e(x',x) + \int_{M^2} f_r(x'',x',x)\, L(x'',x')\, G(x'',x')\, dA''(x'')$$

Integrate over all surfaces

Geometry term

$$G(x'',x') = \frac{\cos\theta_i'' \cos\theta_o'}{\left\| x'' - x' \right\|^2} V(x'',x')$$

Visibility term

$$V(x'',x') = \begin{cases} 1 & \text{visible} \\ 0 & \text{not visible} \end{cases}$$

The Radiosity Equation

Assume diffuse reflection

1. $f_r(x,\omega_i \to \omega_o) = f_r(x) \Rightarrow \rho(x) = \pi f_r(x)$
2. $L(x,\omega) = B(x)/\pi$

$$B(x) = B_e(x) + \rho(x)E(x)$$

$$B(x) = B_e(x) + \rho(x)\int_{M^2} F(x,x')B(x')\,dA'(x')$$

$$\Uparrow$$

$$F(x,x') = \frac{G(x,x')}{\pi}$$

Integral Equations

Integral equations of the 1st kind

$$f(x) = \int k(x,x')g(x')\,dx'$$

Integral equations of the 2nd kind

$$f(x) = g(x) + \int k(x,x')f(x')\,dx'$$

Linear Operators

Linear operators act on functions

like matrices act on vectors

$$h(x) = (L \circ f)(x)$$

They are linear in that

$$L \circ (af + bg) = a(L \circ f) + b(L \circ g)$$

Types of linear operators

$$(K \circ f)(x) \equiv \int k(x, x') f(x')\, dx'$$

$$(D \circ f)(x) \equiv \frac{\partial f}{\partial x}(x)$$

Rendering Operators

Scattering operator

$$L_o(x,\omega_o) = \int_{H^2} f_r(x,\omega_i \to \omega_o) L_i(x,\omega_i) \cos\theta_i\, d\omega_i$$

$$\equiv S \circ L_i$$

Transport transport

$$L_i(x,\omega_i) = L_o(x^*(x,\omega_i), -\omega_i)$$

$$\equiv T \circ L_o$$

Solving the Rendering Equation

Rendering Equation

$$K \equiv S \circ T$$

$$L = L_e + K \circ L$$

$$(I - K) \circ L = L_e$$

Solution

$$L = \left(I - K\right)^{-1} \circ L_e$$

Formal Solution

Neumann series

$$\left(I - K\right)^{-1} = \frac{1}{I - K} = I + K + K^2 + \ldots$$

Verify

$$\begin{aligned}\left(I - K\right) \circ \left(I - K\right)^{-1} &= \left(I - K\right) \circ \left(I + K + K^2 + \ldots\right) \\ &= \left(I + K + \ldots\right) - \left(K + K^2 + \ldots\right) \\ &= I\end{aligned}$$

Successive Approximations

Successive approximations

$$L^1 = L_e$$

$$L^2 = L_e + K \circ L^1$$

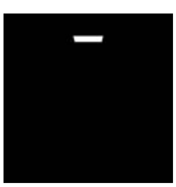
$$\cdots$$

$$L^n = L_e + K \circ L^{n-1}$$

Converged

$$L^n = L^{n-1} \quad \therefore \quad L^n = L_e + K \circ L^n$$

Successive Approximation

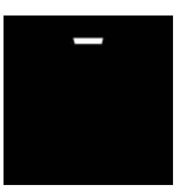


L_e

$K \circ L_e$

$K \circ K \circ L_e$

$K \circ K \circ K \circ L_e$



L_e

$L_e + K \circ L_e$

$L_e + \cdots K^2 \circ L_e$

$L_e + \cdots K^3 \circ L_e$

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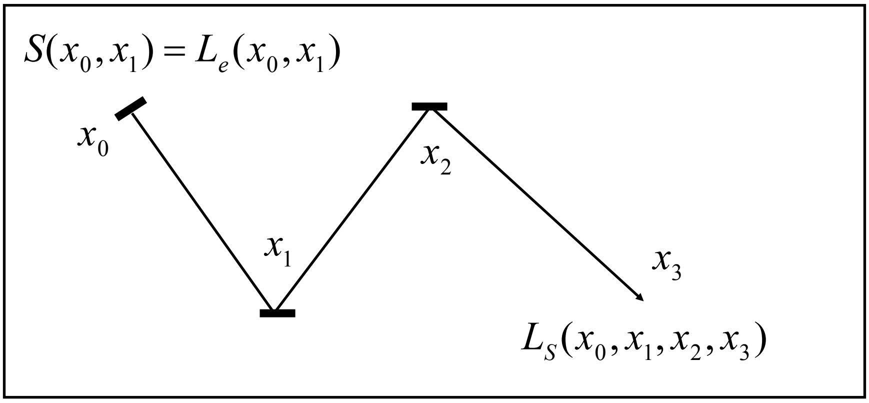
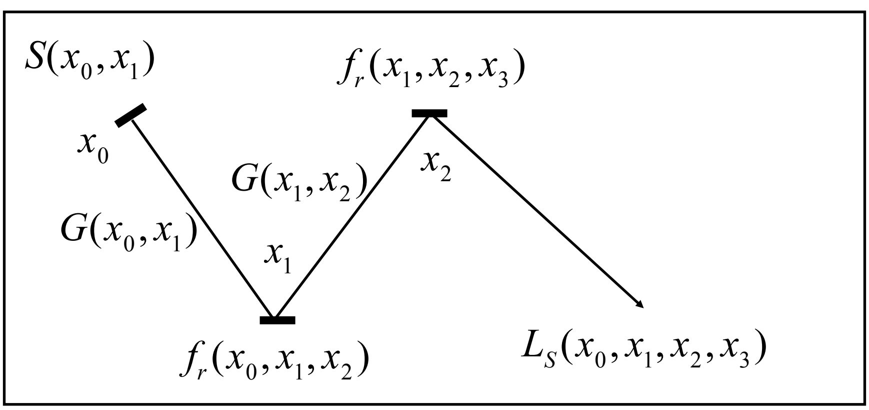
Light Path

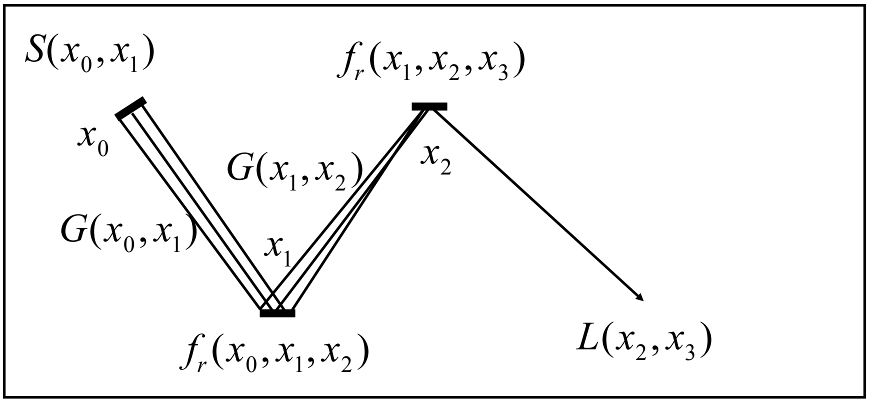


Light Path



$$L_S(x_0,x_1,x_2,x_3) = S(x_0,x_1)G(x_o,x_1)f_r(x_0,x_1,x_2)G(x_1,x_2)f_r(x_1,x_2,x_3)$$

Light Paths



$$L(x_2, x_3) = \int_{A_0}\int_{A_1} L_S(x_0, x_1, x_2, x_3)\, dA(x_0)\, dA(x_1)$$

Light Transport

Integrate over all paths of all lengths

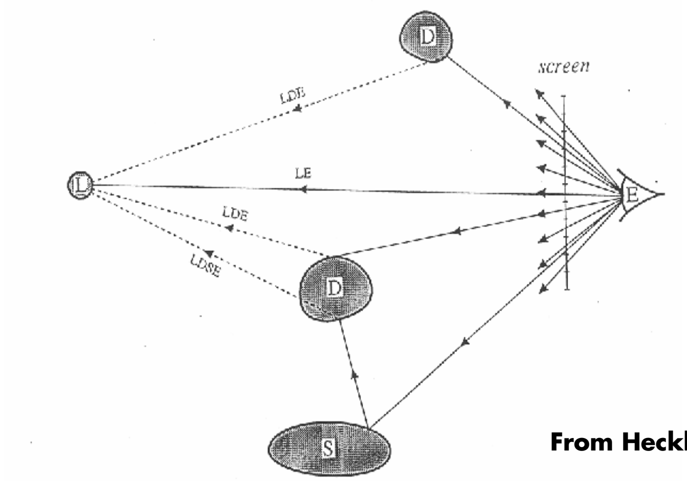
$$L(x_{k-1}, x_k)$$

$$= \sum_{k=1}^{\infty} \int_{M^2} \cdots \int_{M^2} L_S(x_0, \cdots, x_{k-2}, x_{k-1}, x_k)\, dA(x_0) \cdots dA(x_{k-2})$$

Question:

- **How to sample space of paths?**

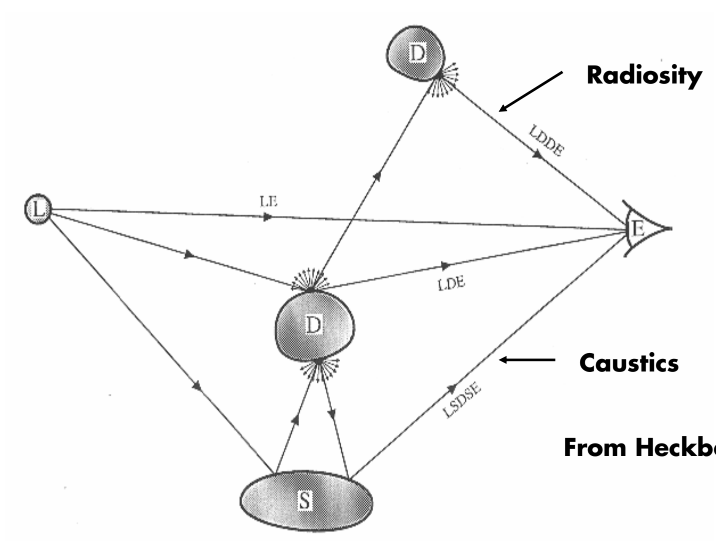
Classic Ray Tracing



From Heckbert

Forward (from eye): E S* (D|G) L

Photon Paths



Radiosity

Caustics

From Heckbert

How to Solve It?

Finite element methods

- Classic radiosity
  - Mesh surfaces
  - Piecewise constant basis functions
  - Solve matrix equation
- Not practical for rendering equation

Monte Carlo methods

- Path tracing (distributed ray tracing)
  - Randomly trace ray from the eye
- Bidirectional ray tracing
- Photon mapping