The Rendering Equation

Direct (*local*) illumination

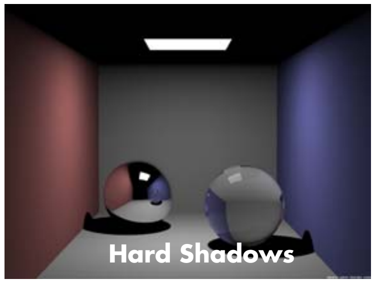
- **Light directly from light sources**
- **No shadows**

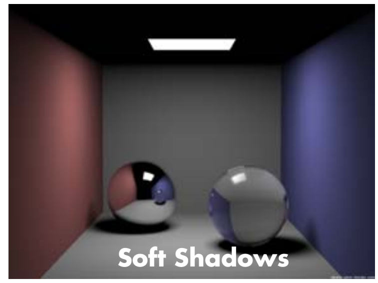
Indirect (*global*) illumination

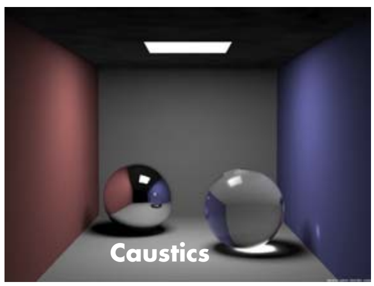
- **Hard and soft shadows**
- **Diffuse interreflections (radiosity)**
- **Glossy interreflections (caustics)**

Early Radiosity

Lighting Effects



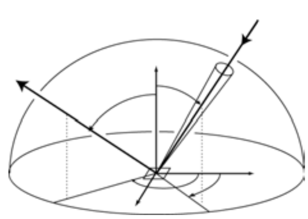




Challenge

To evaluate the reflection equation
the incoming radiance must be known

$$L_r(x,\omega_r) = \int_{H^2} f_r(x,\omega_i \to \omega_r) L_i(x,\omega_i) \cos\theta_i \, d\omega_i$$



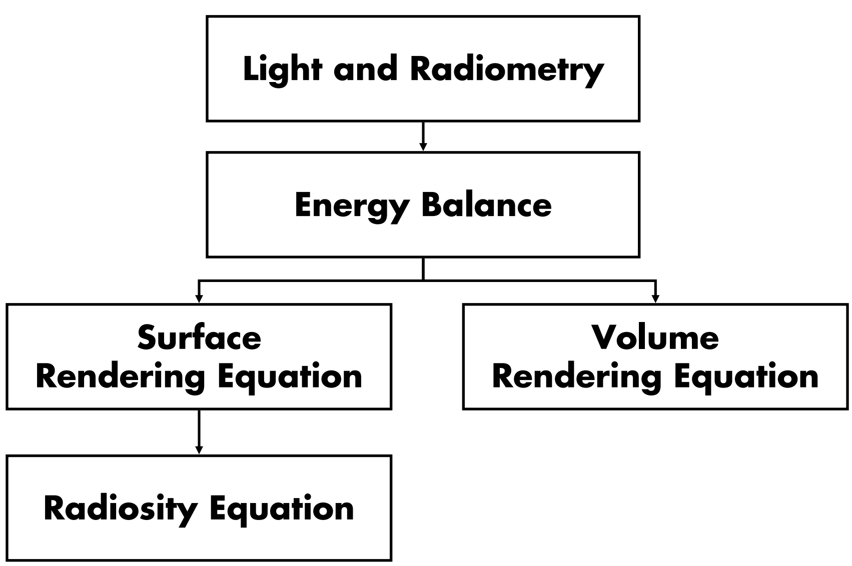
To evaluate the incoming radiance
the reflected radiance must be known

To The Rendering Equation

Questions

1. How is light measured?
2. How is the spatial distribution of light energy described?
3. How is reflection from a surface characterized?
4. What are the conditions for equilibrium flow of light in an environment?

The Grand Scheme



Balance Equation

Accountability

[*outgoing*] - [*incoming*] = [*emitted*] - [*absorbed*]

- **Macro level**

  The total light energy put into the system must equal the energy leaving the system (usually, via heat).

$$\Phi_o - \Phi_i = \Phi_e - \Phi_a$$

- **Micro level**

  The energy flowing into a small region of phase space must equal the energy flowing out.

$$B(x) - E(x) = B_e(x) - E_a(x)$$

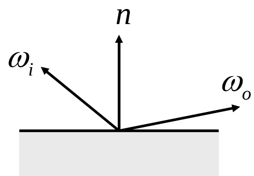
$$L_o(x,\omega) - L_i(x,\omega) = L_e(x,\omega) - L_a(x,\omega)$$

Surface Balance Equation

[*outgoing*] = [*emitted*] + [*reflected*]

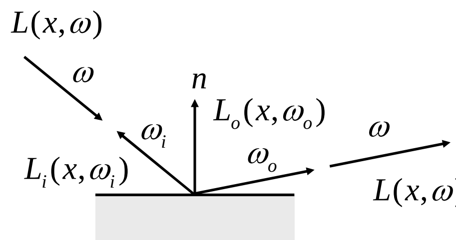
$$\begin{aligned} L_o(x,\omega_o) &= L_e(x,\omega_o) + L_r(x,\omega_o) \\ &= L_e(x,\omega_o) + \int_{H^2} f_r(x,\omega_i \to \omega_o) L_i(x,\omega_i)\cos\theta_i \, d\omega_i \end{aligned}$$

Direction Conventions



$f_r(\omega_i \to \omega_o)$

BRDF



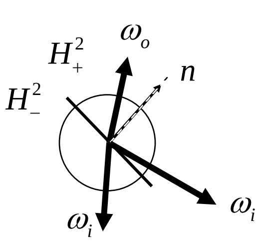
$$L_i(x,\omega_i) = L(x,-\omega_i)$$
$$L(x,\omega) = L_o(x,\omega_o)$$

Surface vs. Field Radiance

Surface Balance Equation

[*outgoing*] = [*emitted*] + [*reflected*] + [*transmitted*]

$$L_o(x,\omega_o) = L_e(x,\omega_o) + L_r(x,\omega_o) + L_t(x,\omega_o)$$



$$L_r(x,\omega_o) = \int_{H_+^2} f_r(x,\omega_i \to \omega_o) L_i(x,\omega_i) \cos\theta_i \, d\omega_i$$

$$L_t(x,\omega_o) = \int_{H_-^2} f_t(x,\omega_t \to \omega_o) L_i(x,\omega_t) \cos\theta_t \, d\omega_t$$

BTDF

$H_+^2(n) \quad \omega_o \bullet n(x) > 0$

$H_-^2(n) \quad \omega_o \bullet n(x) < 0$

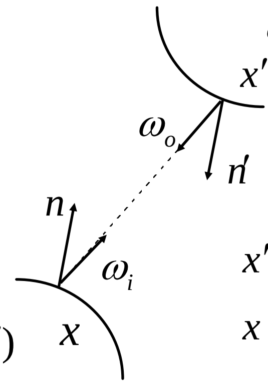
Two-Point Geometry

$$\omega(x,x') = \omega(x \rightarrow x') = \frac{x'-x}{\left|x'-x\right|}$$

Ray Tracing

$$x^*(x,\omega)$$

$$\omega_o = \omega(x',x)$$



$$x' = x^*(x,\omega_i)$$

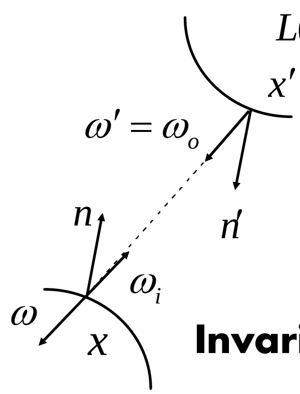
$$x = x^*(x',\omega_o)$$

$$\omega_i = \omega(x,x')$$

Coupling Equations

$$L(x',\omega') = L_o(x',\omega_o)$$

$$\omega' = \omega_o$$



Invariance of radiance

$$L_i(x,\omega_i) = L(x,-\omega_i)$$

$$L(x,\omega) = L(x',\omega')$$

The Rendering Equation

Directional form

$$L(x,\omega) = L_e(x,\omega) + \int_{H^2} f_r(x,\omega' \to \omega)\, L(x^*(x,\omega'), -\omega') \cos\theta'\, d\omega'$$

Integrate over hemisphere of directions

Transport operator i.e. ray tracing

The Rendering Equation

Surface form

$$L(x',x) = L_e(x',x) + \int_{M^2} f_r(x'',x',x)\, L(x'',x')\, G(x'',x')\, dA''(x'')$$

Geometry term

Integrate over all surfaces

$$G(x'',x') = \frac{\cos\theta_i''\cos\theta_o'}{\left\| x''-x' \right\|^2} V(x'',x')$$

Visibility term

$$V(x'',x') = \begin{cases} 1 & \text{visible} \\ 0 & \text{not visible} \end{cases}$$

The Radiosity Equation

Assume diffuse reflection

1. $f_r(x,\omega_i \to \omega_o) = f_r(x) \Rightarrow \rho(x) = \pi f_r(x)$
2. $L(x,\omega) = B(x)/\pi$

$$B(x) = B_e(x) + \rho(x)E(x)$$

$$B(x) = B_e(x) + \rho(x)\int_{M^2} F(x,x')B(x')\,dA'(x')$$

$$F(x,x') = \frac{G(x,x')}{\pi}$$

Integral Equations

Integral equations of the 1st kind

$$f(x) = \int k(x,x')g(x')\,dx'$$

Integral equations of the 2nd kind

$$f(x) = g(x) + \int k(x,x')f(x')\,dx'$$

Linear Operators

Linear operators act on functions like matrices act on vectors

$$h(x) = (L \circ f)(x)$$

They are linear in that

$$L \circ (af + bg) = a(L \circ f) + b(L \circ g)$$

Types of linear operators

$$(K \circ f)(x) \equiv \int k(x, x') f(x') \, dx'$$

$$(D \circ f)(x) \equiv \frac{\partial f}{\partial x}(x)$$

Rendering Operators

Scattering operator

$$L_o(x, \omega_o) = \int_{H^2} f_r(x, \omega_i \to \omega_o) L_i(x, \omega_i) \cos\theta_i \, d\omega_i$$

$$\equiv S \circ L_i$$

Transport transport

$$L_i(x, \omega_i) = L_o(x^*(x, \omega_i), -\omega_i)$$

$$\equiv T \circ L_o$$

Solving the Rendering Equation

Rendering Equation

$$K \equiv S \circ T$$

$$L = L_e + K \circ L$$

$$(I - K) \circ L = L_e$$

Solution

$$L = \left(I - K\right)^{-1} \circ L_e$$

Formal Solution

Neumann series

$$\left(I - K\right)^{-1} = \frac{1}{I - K} = I + K + K^2 + \ldots$$

Verify

$$\begin{aligned}\left(I - K\right) \circ \left(I - K\right)^{-1} &= \left(I - K\right) \circ \left(I + K + K^2 + \ldots\right) \\ &= \left(I + K + \ldots\right) - \left(K + K^2 + \ldots\right) \\ &= I\end{aligned}$$

Successive Approximations

Successive approximations

$$L^1 = L_e$$

$$L^2 = L_e + K \circ L^1$$

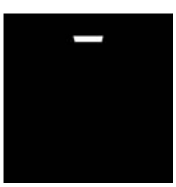
$$\ldots$$

$$L^n = L_e + K \circ L^{n-1}$$

Converged

$$L^n = L^{n-1} \quad \therefore \quad L^n = L_e + K \circ L^n$$

Successive Approximation

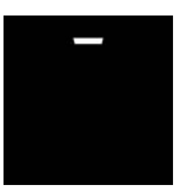


L_e

$K \circ L_e$

$K \circ K \circ L_e$

$K \circ K \circ K \circ L_e$



L_e

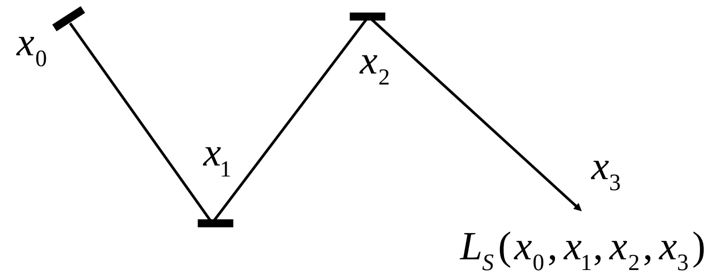
$L_e + K \circ L_e$

$L_e + \cdots K^2 \circ L_e$

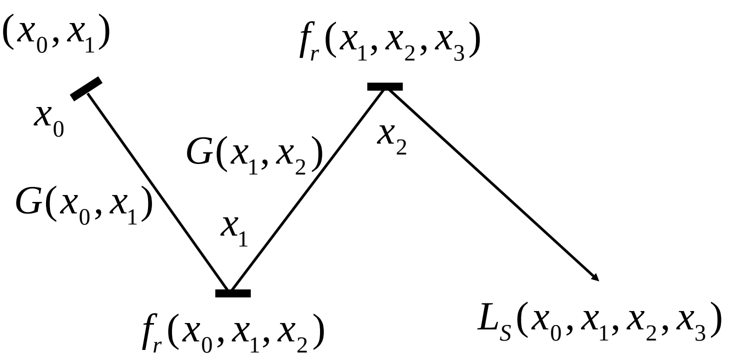
$L_e + \cdots K^3 \circ L_e$

Light Path

$$S(x_0, x_1) = L_e(x_0, x_1)$$

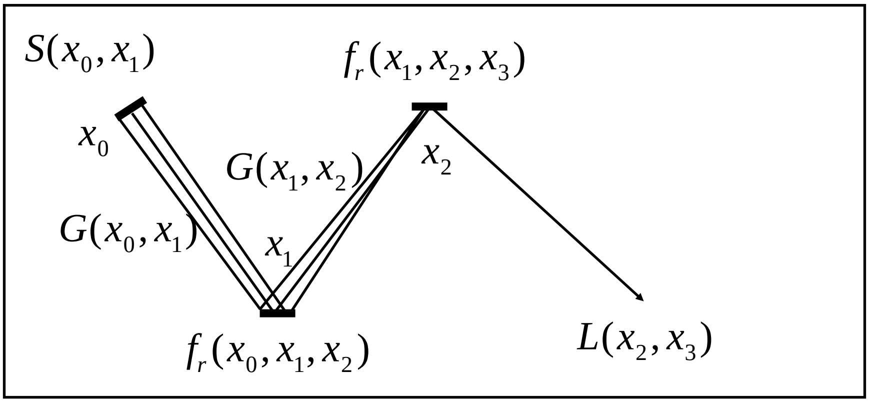


Light Path



$$L_S(x_0, x_1, x_2, x_3) = S(x_0, x_1) G(x_o, x_1) f_r(x_0, x_1, x_2) G(x_1, x_2) f_r(x_1, x_2, x_3)$$

Light Paths



$$L(x_2, x_3) = \int_{A_0} \int_{A_1} L_S(x_0, x_1, x_2, x_3) dA(x_0) dA(x_1)$$

Light Transport

Integrate over all paths of all lengths

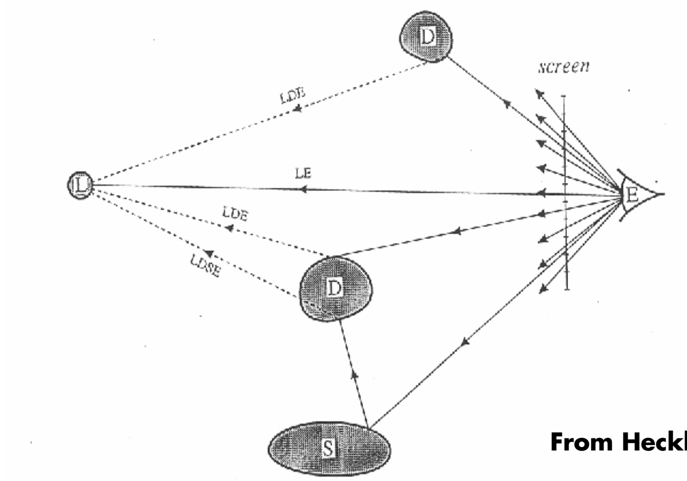
$$L(x_{k-1}, x_k)$$

$$= \sum_{k=1}^{\infty} \int_{M^2} \cdots \int_{M^2} L_S(x_0, \cdots, x_{k-2}, x_{k-1}, x_k) dA(x_0) \cdots dA(x_{k-2})$$

Question:

- **How to sample space of paths?**

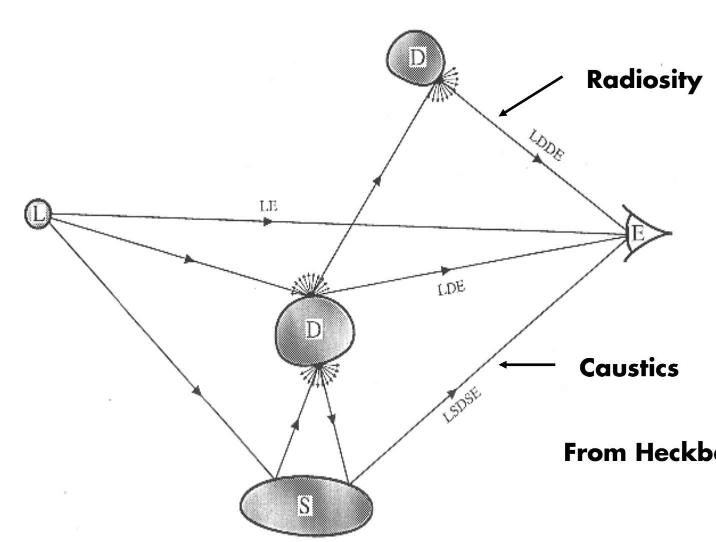
Classic Ray Tracing



From Heckbert

Forward (from eye): E S* (D|G) L

Photon Paths



Radiosity

Caustics

From Heckbert

How to Solve It?

Finite element methods

- Classic radiosity
  - Mesh surfaces
  - Piecewise constant basis functions
  - Solve matrix equation
- Not practical for rendering equation

Monte Carlo methods

- Path tracing (distributed ray tracing)
  - Randomly trace ray from the eye
- Bidirectional ray tracing
- Photon mapping