# Monte Carlo I

**Previous lecture**

- **Analytical illumination formula**

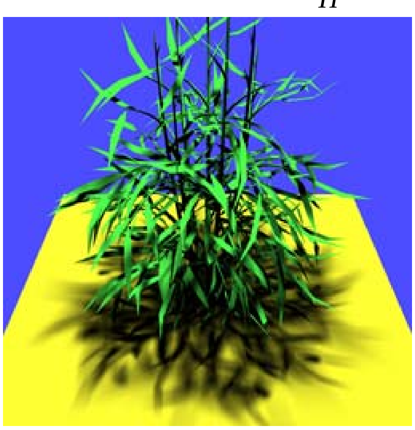
**This lecture**

- **Numerical evaluation of illumination**
- **Review random variables and probability**
- **Monte Carlo integration**
- **Sampling from distributions**
- **Sampling from shapes**
- **Variance and efficiency**



# Lighting and Soft Shadows

$$E(x) = \int_{H^2} L_i(x,\omega) \cos\theta \, d\omega$$

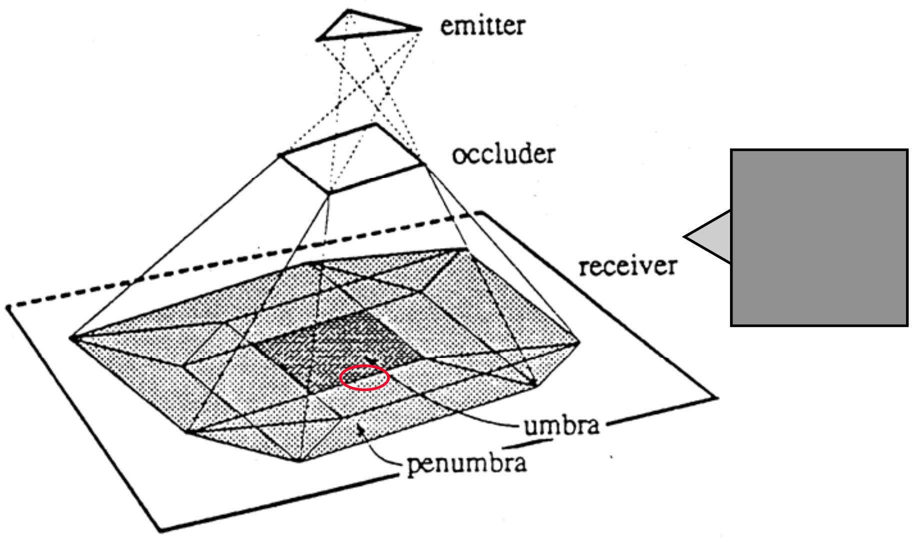


**Challenges**

- **Visibility and blockers**
- **Varying light distribution**
- **Complex source geometry**

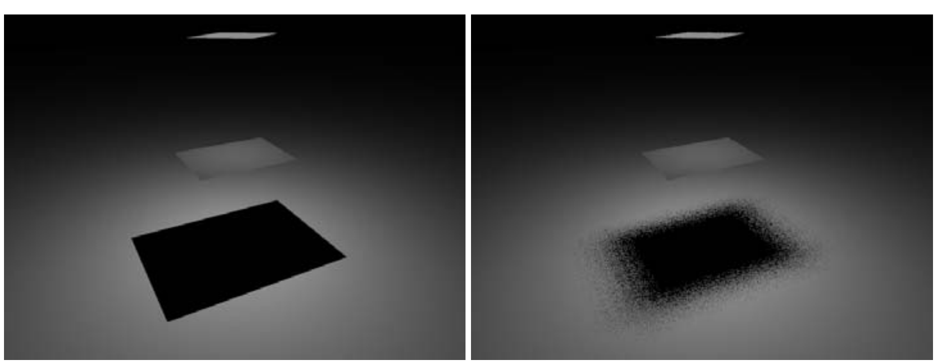
**Source: Agrawala. Ramamoorthi, Heirich, Moll, 2000**

## Penumbras and Umbras






## Monte Carlo Lighting



**4 eye rays per pixel**
**1 shadow ray per eye ray**

**Fixed** **Random**

## Monte Carlo Algorithms

**Advantages**

- Easy to implement
- Easy to think about (but be careful of statistical bias)
- Robust when used with complex integrands and domains (shapes, lights, ...)
- Efficient for high dimensional integrals
- Efficient solution method for a few selected points

**Disadvantages**

- Noisy
- Slow (many samples needed for convergence)



## Random Variables

$X$ is chosen by some random process

$X \sim p(x)$ probability distribution function (PDF)

## Discrete Probability Distributions

**Discrete events $X_i$ with probability $p_i$**

$$p_i \geq 0 \qquad \sum_{i=1}^{n} p_i = 1$$

**Cumulative PDF**

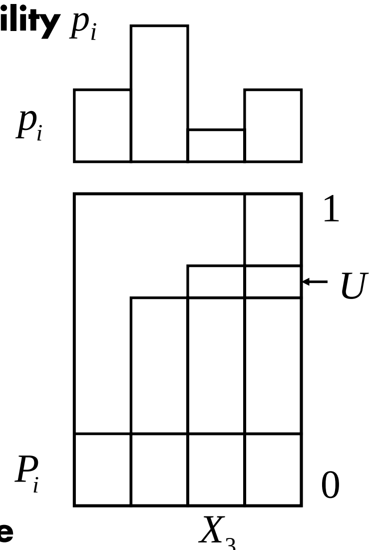
$$P_j = \sum_{i=1}^{j} p_i$$

**Construction of samples**

**To randomly select an event,**

**Select $X_i$ if $P_{i-1} < U \leq P_i$**

**Uniform random variable**






## Continuous Probability Distributions

**PDF** $p(x)$

$$p(x) \geq 0$$

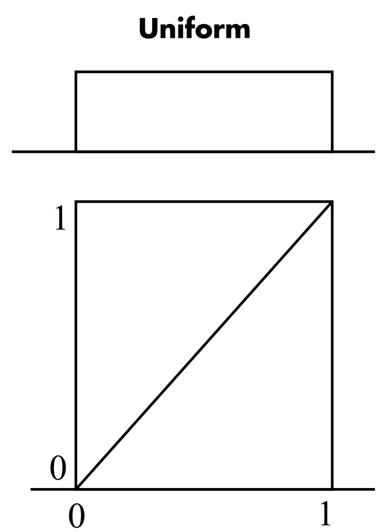
**CDF** $P(x)$

$$P(x) = \int_0^x p(x)dx$$

$$P(x) = \Pr(X < x) \qquad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$



## Sampling Continuous Distributions

**Cumulative probability distribution function**

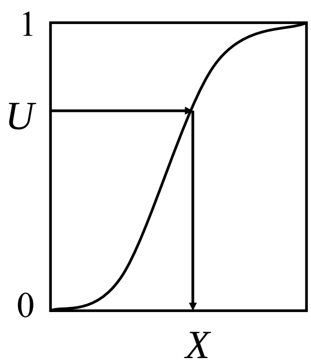
$$P(x) = \Pr(X < x)$$

**Construction of samples**

**Solve for** $X = P^{-1}(U)$

**Must know:**

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$






## Example: Power Function

**Assume**

$$p(x) = (n+1)x^n$$

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

**Trick**

$$Y = \max(U_1, U_2, \cdots, U_n, U_{n+1})$$

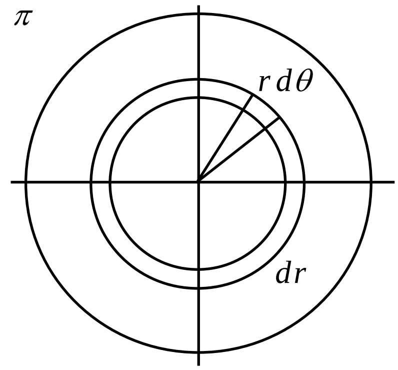
$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U < x) = x^{n+1}$$

## Sampling a Circle

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \left(\frac{r^2}{2}\right)\Bigg|_0^1 \theta\Big|_0^{2\pi} = \pi$$

$$p(r,\theta)\,dr\,d\theta = \frac{1}{\pi} r\,dr\ d\theta \Rightarrow p(r,\theta) = \frac{r}{\pi}$$

$$p(r,\theta) = p(r)\,p(\theta)$$




$$p(\theta) = \frac{1}{2\pi}$$

$$P(\theta) = \frac{1}{2\pi}\theta$$

$$\theta = 2\pi U_1$$

$$p(r) = 2r$$

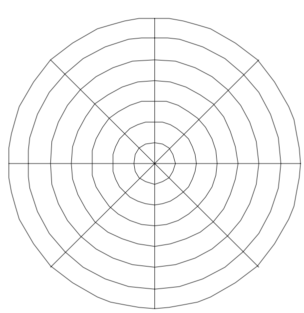
$$P(r) = r^2$$

$$r = \sqrt{U_2}$$



## Sampling a Circle

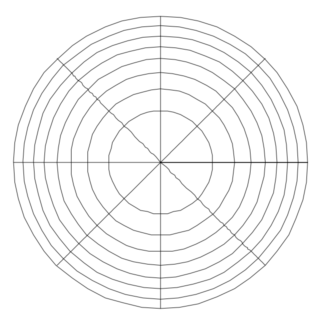
**WRONG $\neq$ Equi-Areal**



$$\theta = 2\pi U_1$$

$$r = U_2$$

**RIGHT = Equi-Areal**



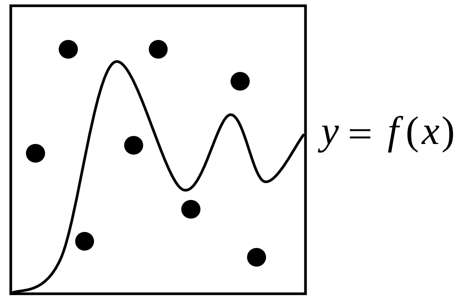
$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

## Rejection Methods

$$I = \int_0^1 f(x)\,dx$$

$$= \iint_{y<f(x)} dx\,dy$$




**Algorithm**

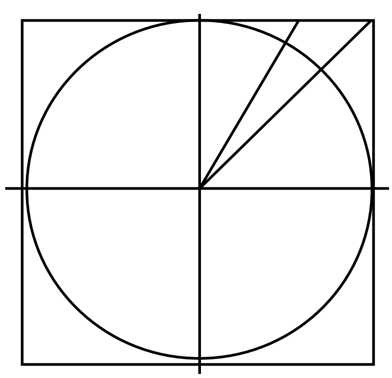
**Pick** $U_1$ **and** $U_2$

**Accept** $U_1$ **if** $U_2 < f(U_1)$

**Wasteful? Efficiency = Area / Area of rectangle**



## Sampling a Circle: Rejection



```
do {
   X=1-2*U1
   Y=1-2*U2
while( X2+ Y2 >1 )
```

$X = 1 - 2 U_1$, $Y = 1 - 2 U_2$, while( $X^2 + Y^2 > 1$ )

**May be used to pick random 2D directions**

**Circle techniques may also be applied to the sphere**

## Monte Carlo Integration

**Definite integral** $I(f) \equiv \int_0^1 f(x)\,dx$

**Expectation of *f*** $E[f] \equiv \int_0^1 f(x)\,p(x)\,dx$

**Random variables** $X_i \sim p(x)$

$Y_i = f(X_i)$

**Estimator** $F_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$



## Unbiased Estimator

$$\boxed{E[F_N] = I(f)}$$

$$E[F_N] = E[\frac{1}{N}\sum_{i=1}^{N} Y_i]$$
$$= \frac{1}{N}\sum_{i=1}^{N} E[Y_i] = \frac{1}{N}\sum_{i=1}^{N} E[f(X_i)]$$
$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)p(x)\,dx$$
$$= \frac{1}{N}\sum_{i=1}^{N}\int_0^1 f(x)dx$$
$$= \int_0^1 f(x)\,dx$$

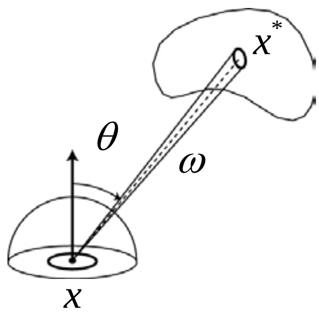
**Assume uniform probability distribution for now**

**Properties**

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$
$$E[aY] = aE[Y]$$

## Direct Lighting – Directional Sampling

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta \, d\omega$$




**Ray intersection** $x^*(x,\omega)$

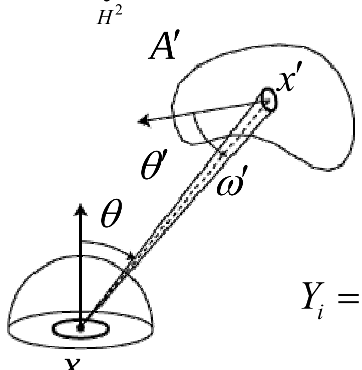
**Sample** $\omega$ **uniformly by** $\Omega$

$$Y_i = L(x^*(x,\omega_i), -\omega_i)\cos\theta_i \; 2\pi$$



## Direct Lighting – Area Sampling

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta \, d\omega = \int_{A'} L_o(x',\omega')V(x,x')\frac{\cos\theta\cos\theta'}{\left|x-x'\right|^2}dA'$$



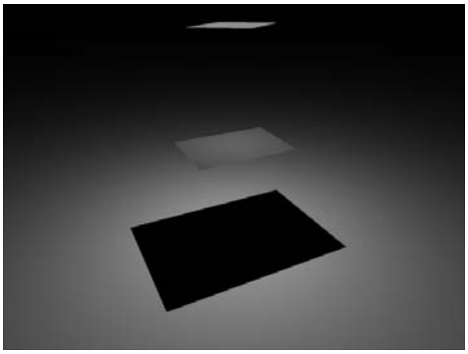

**Ray direction** $\omega' = x - x'$

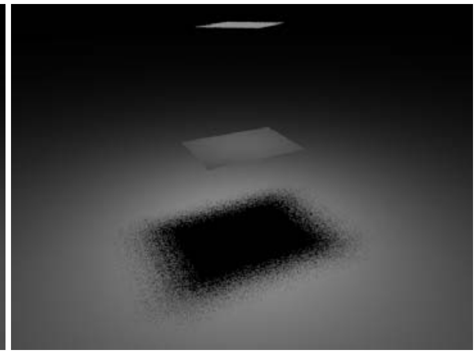
**Sample** $x'$ **uniformly by** $A'$

$$Y_i = L_o(x_i',\omega_i')\,V(x,x_i')\frac{\cos\theta_i\,\cos\theta_i'}{\left|x-x_i'\right|^2}A$$

$$V(x,x') = \begin{cases} 0 & \neg visible \\ 1 & visible \end{cases}$$

## Examples



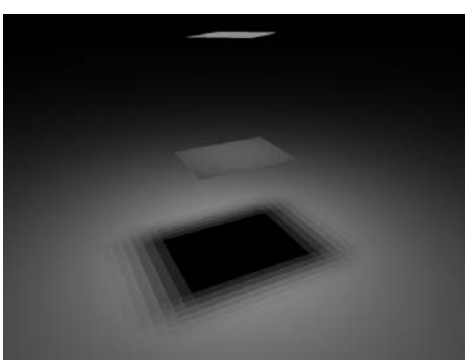


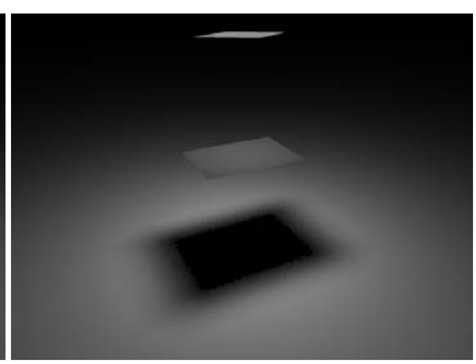
**4 eye rays per pixel**
**1 shadow ray per eye ray**

**Fixed** | **Random**



## Examples

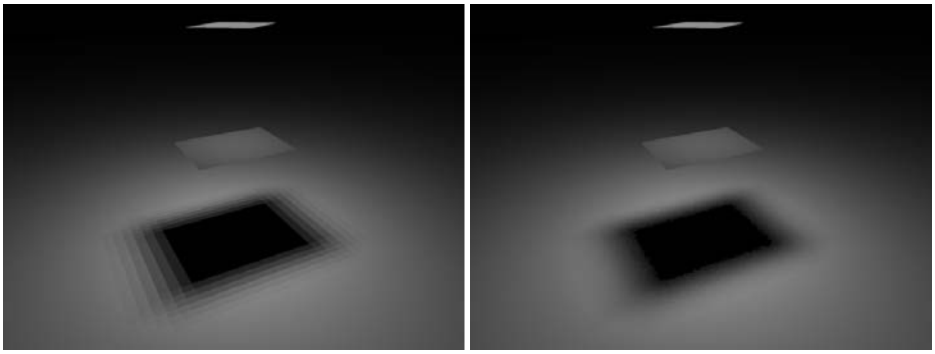




**4 eye rays per pixel**
**16 shadow rays per eye ray**

**Uniform grid** | **Stratified random**

## Examples

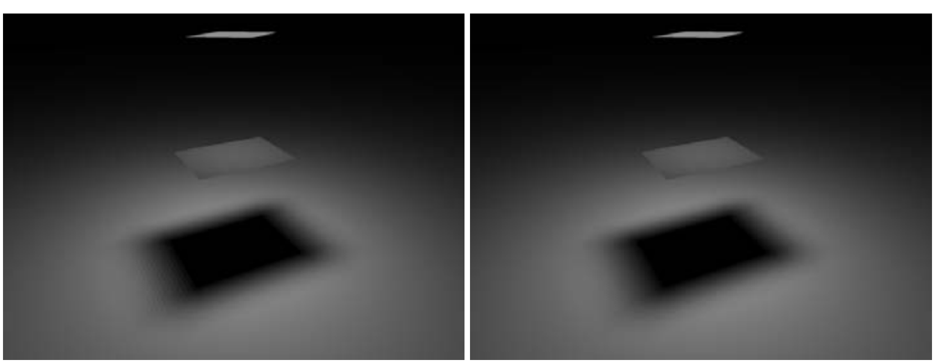


**4 eye rays per pixel**
**64 shadow rays per eye ray**

**Uniform grid** **Stratified random**



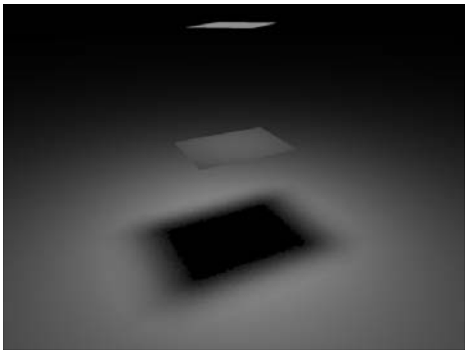
## Examples

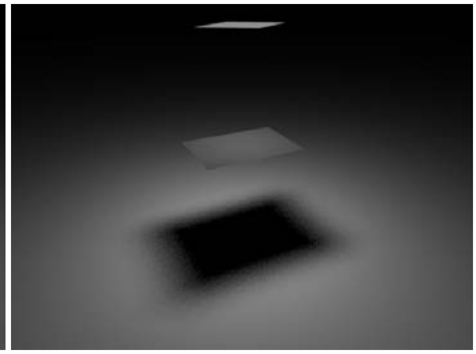


**4 eye rays per pixel**
**100 shadow rays per eye ray**

**Uniform grid** **Stratified random**

## Examples





4 eye rays per pixel
16 shadow rays per eye ray

64 eye rays per pixel
1 shadow ray per eye ray



## Variance

**Definition**

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2 - 2YE[Y] + E[Y]^2]$$
$$= E[Y^2] - E[Y]^2$$

**Properties**

$$V[\sum_i Y_i] = \sum_i V[Y_i]$$

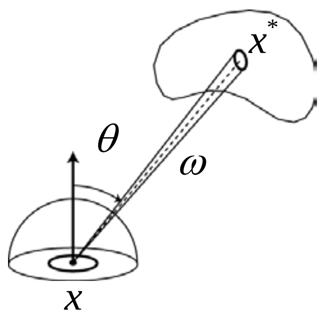
$$V[aY] = a^2 V[Y]$$

**Variance decreases with sample size**

$$V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N} V[Y]$$

## Direct Lighting – Directional Sampling

$$E(x) = \int_{H^2} L(x,\omega)\cos\theta d\omega$$




**Ray intersection** $x^*(x,\omega)$

**Sample** $\omega$ **uniformly by** $\Omega$

$$Y_i = L(x^*(x,\omega_i),-\omega_i)\cos\theta_i\ 2\pi$$

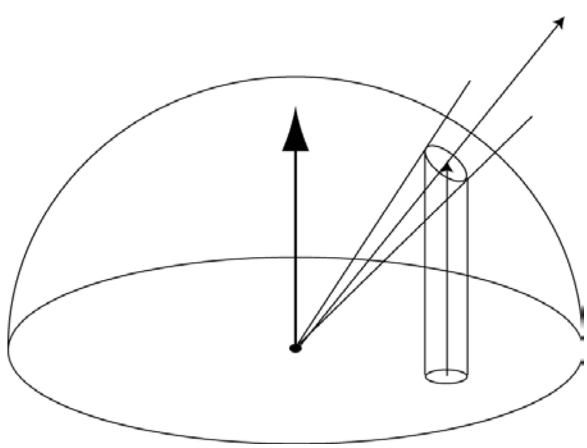
**Sample** $\omega$ **uniformly by** $\tilde{\Omega}$

$$Y_i = L(x^*(x,\omega_i),-\omega_i)\ \pi$$

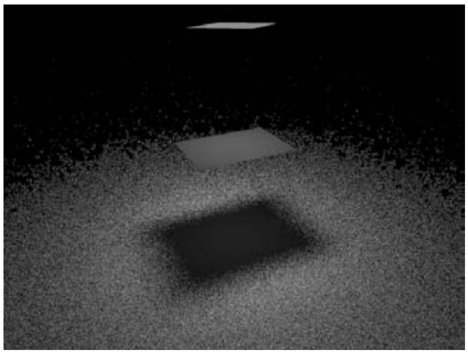
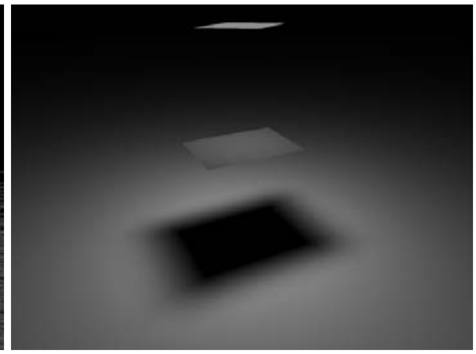


## Sampling Projected Solid Angle

**Generate cosine weighted distribution**



## Examples

|  |  |
|---|---|
| **Projected solid angle** | **Area** |
| **4 eye rays per pixel**<br>**100 shadow rays** | **4 eye rays per pixel**<br>**100 shadow rays** |



## Variance Reduction

**Efficiency measure**

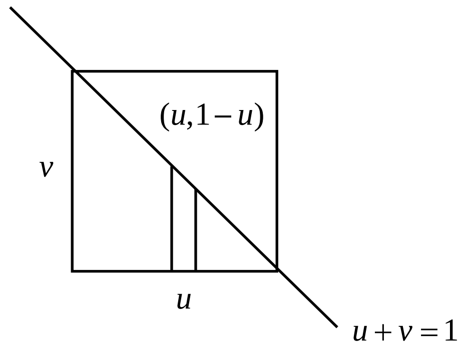
$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

**Techniques**

- **Importance sampling**
- **Sampling patterns: stratified, ...**

## Sampling a Triangle

$$u \ge 0$$
$$v \ge 0$$
$$u + v \le 1$$




$$A = \int_0^1 \int_0^{1-u} dv\, du = \int_0^1 (1-u)\, du = -\frac{(1-u)^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$



## Sampling a Triangle

**Here *u* and *v* are not independent!** $p(u,v) = 2$

**Conditional probability**

$$p(u) \equiv \int p(u,v)\, dv \qquad p(u \mid v) \equiv \frac{p(u,v)}{p(u)}$$

$$p(u) = 2\int_0^{1-u} dv = 2(1-u)$$

$$u_0 = 1 - \sqrt{U_1}$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\, du \quad = (1-u_0)^2$$

$$v_0 = \sqrt{U_1} U_2$$

$$p(v \mid u) = \frac{1}{(1-u)}$$

$$P(v_0 \mid u_0) = \int_0^{v_o} p(v \mid u_0)\, dv \quad = \int_0^{v_o} \frac{1}{(1-u_0)} dv \quad = \frac{v_0}{(1-u_0)}$$