Developing an Algorithm for Visualizing Metamaterials

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Introduction

For my final project, I was interested in rendering optical metamaterials. Metamaterials are synthetic materials which can be engineered to have properties not found in nature. The field of transformation optics is concerned with designing such metamaterials whose microstructure manipulates the propagation of elecromagnetic waves in order to achieve a specific optical effect, in many cases a coordinate transformation. A good summary of the theory of the field is available in an overview by Leonhardt et al. [1].

Some of the theoretical effects which can be achieved using materials include perfect cloaking devices, pictured in figure 1, where light bends around an interior cavity of radius R_1. Other theoretical possibilities include negative refraction materials, and perfect lenses. Although macro-scale implementations of these types of metamaterials are still out of reach of today's technology, primarily due to the difficultly in manufacturing with the nanoscale precision needed for optics, devising a general purpose algorithm for visualizing these materials presents an interesting computational challenge, which has not yet been addressed by the literature. Here, I will describe an algorithm I developed as a final project for CS348b: Image Synthesis and display proof-of-concept images which show that the rendering algorithm I have developed is viable.

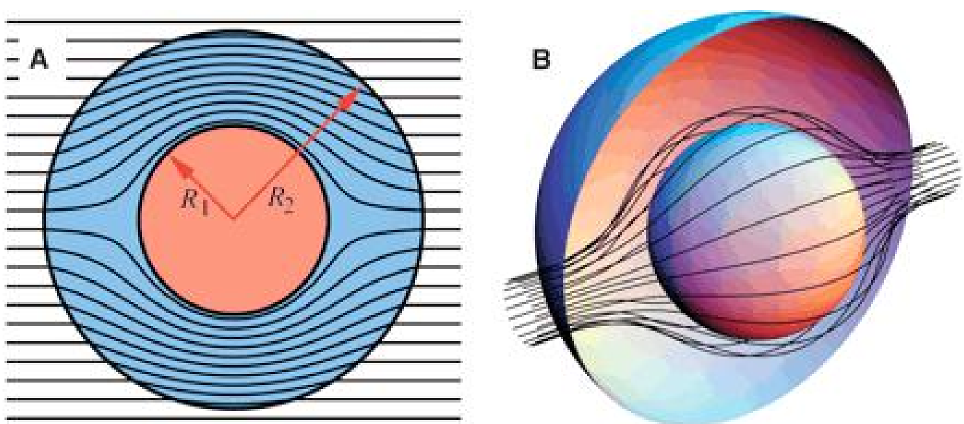

Figure 1: A perfect cloaking device wherein light bends around the interior cavity of radius R_1.

The Geometry of Light

To be more precise about how optical metamaterials differ from normal materials, in traditional ray optics, paths of light γ follow the Principle of Least Action, extremizing the Lagrangian

$$\mathcal{L}[\gamma] = \int_\gamma \eta \, ds = \int_\gamma \eta \sqrt{dx^2 + dy^2 + dz^2} \,, \tag{1}$$

where $\eta(x)$ is the index of refraction of a material. One can think of the light as travelling through a isotropically deformed space, where $\eta(x)$ is the local scaling factor at position x. Light then travels the shortest distance path between the endpoints of γ in this deformed space. The key difference between normal ray optics and transformation optics is that transformation optics allows more general local deformation of space. The distortion is given by a 3D Riemannian metric $g_{\mu\nu}(x)$ which results from the nanoscale properties of the material. Think of $g_{\mu\nu}(x)$ as a 3×3 symmetric positive definite matrix at each position x in space such that $\sqrt{v^T g v}$ is the length of an infinitesimally small vector v at x. The Principle of Least Action then says that the paths of light γ should extremize the Lagrangian

$$\mathcal{L}[\gamma] = \int_\gamma \sqrt{g_{\mu\nu}\, dx^\mu\, dx^\nu}\,, \tag{2}$$

$$\mathcal{L}[\gamma] = \int_\gamma \sqrt{\sum_{\mu=1}^{3}\sum_{\nu=1}^{3} g_{\mu\nu}(x)\, dx^\mu\, dx^\nu}\,, \tag{3}$$

where we are using Einstein notation – there is an implicit sum over μ and ν under the square-root. Applying the Euler-Lagrange equations to $\mathcal{L}$ yields the geodesic equations when the metric $g_{\mu\nu}$ is smooth,

$$\frac{d^2u^\mu}{ds^2} = -\Gamma^\mu{}_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\,, \tag{4}$$

where $u^\mu(s)$ is a parameterization of the curve γ (called a geodesic) in the variable s. Here $\Gamma^\mu{}_{\alpha\beta}(x)$ are the Christoffel symbols of the Levi-Civita connection induced by the metric $g_{\mu\nu}(x)$, defined by

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu m}\left(\partial_\beta g_{m\alpha} + \partial_\alpha g_{m\beta} - \partial_m g_{\alpha\beta}\right) \tag{5}$$

where $g^{\mu\nu}(x)$ is the inverse of $g_{\mu\nu}(x)$ at x. Hence inside an optical metamaterial which implements the smooth metric $g_{\mu\nu}$, light propagates along the paths given by the Geodesic equations (4). A pictorial example can be seen in figure 2. I want to design and implement an algorithm for rendering a metamaterial in a region Ω with prescribed metric $g_{\mu\nu}(x)$.

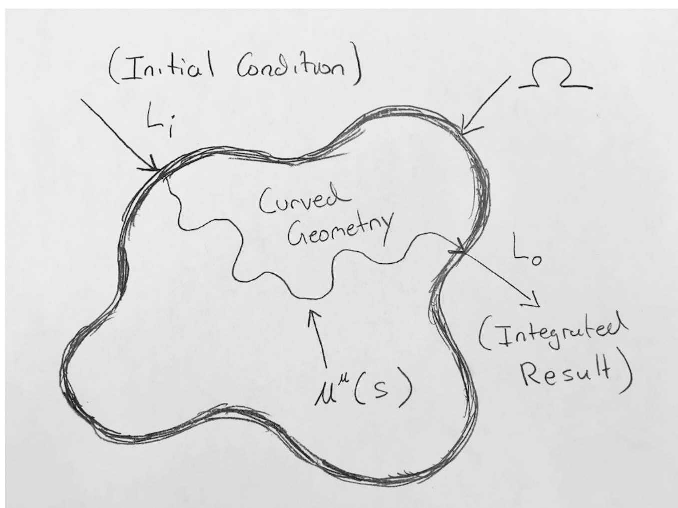

Figure 2: Propagation of a light ray $\mu^\mu(s)$ through a metamaterial.

We note that oftentimes, these metrics derive from certain coordinate transformations, as performing a coordinate transformation can be thought of as "bending space" and has an associated metric $g_{\mu\nu}(x)$ (the pushforward of the Euclidean metric). As an example, consider figure (3). This figure shows a coordinate transformation which implements the perfect cloaking device seen in figure (1). We see that the ambient space is bent in such a way that it distorts the space and guides paths of light around the central cavity, thereby rendering the cavity invisible.

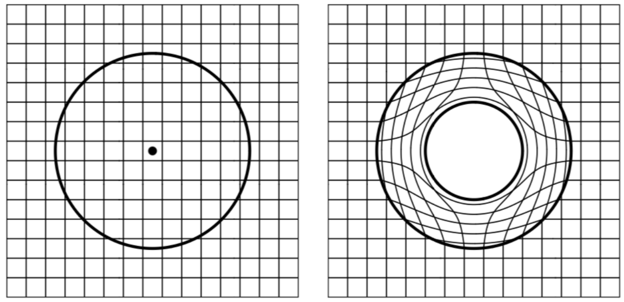

Figure 3: A coordinate transformation which implements the cloaking device, seen in figure (1), note how the geodesics curve around the interior cavity.

Previous Work

There seems to have been some small amount of work in the area of visualizing metamaterials [2, 3, 4], however, the ones I have seen are all for visualizing very specific metamaterials, and none provide a general purpose algorithm. There is also similar existing work on rendering where the index of refraction $\eta(x)$ varies throughout a material, or throughout the atmosphere. One can think of this new problem as a sort of generalization of this, as the former amounts to setting the metric $g_{\mu\nu}(x) = \eta(x)^2\delta_{\mu\nu}$ where $\delta_{\mu\nu}$ is the Euclian metric (i.e. identity matrix). There has also been work done on rendering black holes, most notably in the movie *Interstellar* [5]. The problems are similar, but the work done in [5] revolves around solving for null curves, where $g_{\mu\nu}\, dx^\mu\, dx^\nu = 0$, for a very specific metric $g_{\mu\nu}$, the Kerr metric for rotating black holes. There are also other auxiliary considerations which don't need to be considered in the metamaterial setting, such as red/blue shift from gravitation. Conversely, in the metamaterial setting, there are problems which one doesn't have in the black hole setting, such as how to handle intersection with geometry.

My Proposed Rendering Algorithms

I propose two rendering algorithms for metamaterials with an underlying metric $g_{\mu\nu}$. **Everything contained in the remaining sections is my own original work, and to the best of my knowledge, hasn't been tried before**.

Coordinate Transformed Distance Estimators

In this algorithm, we suppose we want to render a metamaterial in some domain Ω whose metric $g_{\mu\nu}$ comes from some coordinate transformation $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. That is,

$$g_{\mu\nu} = (D\Phi)_\mu{}^\alpha (D\Phi)_\nu{}^\beta \eta_{\alpha\beta} \tag{6}$$

or written in alternate notation,

$$g = (D\Phi)^T (D\Phi) \tag{7}$$

where $D\Phi$ is the Jacobian of the transformation Φ. This is the transformation of the Euclidean metric under Φ. One can see in figure (4) that the preimage $\Phi^{-1}(\Omega)$ has the Euclidean metric, and hence geodesics in $\Phi^{-1}(\Omega)$ are just straight lines. This means that in $\Phi^{-1}(\Omega)$ one can just ray-trace as normal. However, for primitives which are made up of linear elements (i.e. triangles), we note that they may be nonlinear or distorted in $\Phi^{-1}(\Omega)$. Fortunately, there is a very natural way to transform distance estimators.

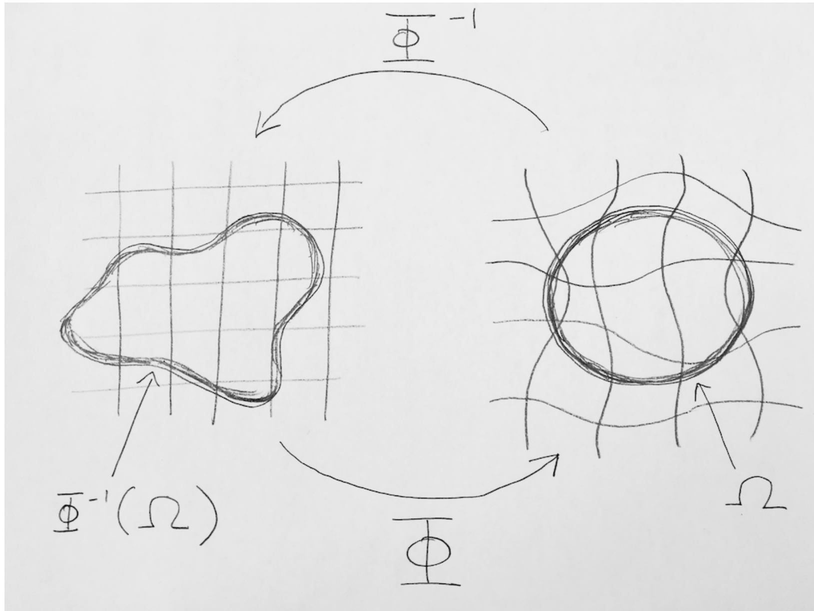

Figure 4: Warping space with a coordinate transformation.

Suppose that we're given a signed distance estimator f on Ω which estimates the distance from x to the boundary $\partial\Omega$ at every point in the interior of Ω. That is, for any point $x \notin \text{int}(\Omega)$ and any point $y \in \partial\Omega$, we have

$$0 < |f(x)| \leq \|x - y\|_2 \,. \tag{8}$$

with $f(x) < 0$ in the interior of Ω and $f(x) > 0$ in the exterior. Then, consider the function $f \circ \Phi$ on $\Phi^{-1}(\Omega)$. For every $y \in \partial\Omega$, we have

$$0 < -f(\Phi(x)) \leq \|\Phi(x) - y\|_2 \,. \tag{9}$$

Since Φ is smooth and bijective, there exists a $z \in \partial\Phi^{-1}(\Omega) = \Phi^{-1}(\partial\Omega)$ such that $y = \Phi(z)$, so

$$0 < -f(\Phi(x)) \leq \|\Phi(x) - \Phi(z)\|_2 \,. \tag{10}$$

Using a Taylor expansion, we have that

$$\|\Phi(x) - \Phi(z)\|_2 = \|D\Phi(x)\, x\|_2 + O(|x|^2) \leq \|D\Phi(x)\|_2 \|x - z\|_2 + O(\|x - z\|_2^2) \,, \tag{11}$$

where $\|D\Phi(x)\|_2$ denotes the operator 2 norm of the Jacobian $D\Phi(x)$. Therefore

$$0 < -\frac{f(\Phi(x))}{\|D\Phi(x)\|_2} \leq \|x - z\|_2 + O(\|x - z\|_2^2) \,. \tag{12}$$

for any $x \in \Phi^{-1}(\Omega)$ and any $z \in \partial\Phi^{-1}(\Omega)$. This means that $h \equiv -(f \circ \Phi)/\|D\Phi\|_2$ is an approximate distance estimator in the preimage $\Phi^{-1}(\Omega)$, as long as the second derivative of Φ is not too large! Therefore, we can do standard ray marching with this distance estimator in $\Phi^{-1}(\Omega)$. The algorithm would look something like this:

1. **Input**: Position $x_i \in \partial\Omega$ and direction ω_i where light has entered the object Ω.

2. Transform x_i and ω_i into the preimage $\partial\Phi^{-1}(\Omega)$:

$$\begin{aligned} x_i' &\leftarrow \Phi^{-1}(x_i) \\ \omega_i' &\leftarrow (D\Phi(x_i))^{-1}\omega_i \end{aligned} \tag{13}$$

3. Ray march using the distance estimator $-(f \circ \Phi)/\|D\Phi\|_2$ until we hit the boundary $\partial\Phi^{-1}(\Omega)$.

4. Transform the resulting intersection position and direction x_o' and ω_o' back to the space Ω:

$$\begin{aligned} x_o &\leftarrow \Phi(x_o') \\ \omega_o &\leftarrow (D\Phi(x_o'))\omega_o' \end{aligned} \tag{14}$$

5. **Output**: light leaves the object Ω at position x_o in direction ω_o.

Note that the computation of $\|D\Phi(x)\|_2$ can be expensive as it may require doing power method or taking an SVD. Instead, one could also use the Frobenius or the geometric average of L_1 and L_∞ operator norms,

$$\|D\Phi\|_2 \leq \|D\Phi\|_F \qquad \|D\Phi\|_2 \leq \sqrt{\|D\Phi\|_1 \|D\Phi\|_\infty}\,, \tag{15}$$

both of which are significantly easier to compute. In my implementation, I use the Frobenius norm.

Note that if one has access to higher derivatives, for example, second order, then one can use a higher order Taylor expansion, to construct an approximate distance estimator, e.x.

$$\|\Phi(x) - \Phi(z)\|_2 \leq \|D\Phi\|_2 \|x - z\|_2 + \frac{1}{2}\|D^2\Phi\|_2 \|x - z\|_2^2 + O(\|x - z\|_2^3) \tag{16}$$

This algorithm has a number of advantages and drawbacks. The first advantage is that it is relatively easy to implement, assuming one has been given the forward and backward coordinate transformations Φ and Φ^{-1}, the Jacobians $D\Phi$ and $D\Phi^{-1}$ (note we only actually need one, since they are inverses of each other), and the signed distance estimator f. The algorithm is also relatively efficient. However, it can only handle metamaterials which derive their underlying metric from a coordinate transformation. This is, of course, only a small subset of possible metamaterials – an important subset – but nonetheless small. Therefore, I also propose the numerical integration of geodesics algorithm below.

Numerical Integration of Geodesics

The alternative method I propose, is to actually solve the geodesic equation (4) outright and compute the path $x^\mu(s)$ the light takes through the material Ω directly. Note that while the previous method works only when $g_{\mu\nu}$ comes from a coordinate transformation Φ and when we have a computable distance estimator f for the boundary of Ω. This method is much more general. We can rewrite the geodesic equations as a system of first-order ODEs,

$$\frac{d}{ds}\begin{bmatrix} u^\mu \\ v^\nu \end{bmatrix} = \begin{bmatrix} v^\mu \\ -\Gamma^\nu{}_{\alpha\beta} v^\alpha v^\beta \end{bmatrix} \tag{17}$$

where $u^\mu(s)$ is the position of the geodesic at s, and $v^\mu(s)$ is the velocity of the geodesic at s. Note that, when a ray of light intersects the domain Ω, we have the initial conditions $u^\mu(0)$ and $v^\mu(0)$ are these are just the position of intersection and ray direction respectively. Therefore, one can numerically integrate the above ODE forward in time until $u^\mu(s)$ hits the boundary $\partial\Omega$. I propose using a Runge-Kutta scheme (perhaps RK4). The full algorithm will look like this,

1. **Input**: Position $x_i \in \partial\Omega$ and direction ω_i where light has entered the object Ω.
2. Write the system (17) as
$$\frac{d\vec{y}}{ds} = f(\vec{y})\,, \tag{18}$$
where $\vec{y}$ is the combined position and velocity of the light ray. Using the initial values x_i and ω_i, integrate the system while checking for intersection with $\partial\Omega$ at every time step. When we finally intersect $\partial\Omega$, let x_o and ω_o be the position and velocity values of $\vec{y}$.
3. **Output**: light leaves the object Ω at position x_o in direction ω_o.

Analogues for Snell's Law and the Law of Reflection

If light crosses a discontinuity in the metric $g_{\mu\nu}$, then it logically bends in some fashion, as is the case when light enters a material with different index of refraction. I was unable to find an expression for the manner in which light bends for metamaterials when it crosses a discontinuity in $g_{\mu\nu}$. So, I derived myself how to calculate refraction and reflection angles using the Principle of Least Action.

Suppose light is passing from a material with metric $g^{(i)}$ to a material with metric $g^{(o)}$. By looking at an infinitesimally small region around the point $y(\alpha, \beta)$ where the light penetrates the boundary, we may assume that $g^{(i)}$ and $g^{(o)}$ are constant on either sides of the boundary. Moreover, since $g^{(i)}$ and $g^{(o)}$ are constant, light travels in a straight line, bending only at the boundary. Again, since we are considering the problem infinitesimally, we may assume that the point the light intersects the boundary is given by $y(\alpha, \beta) = \alpha u + \beta v$, where u and v are two arbitrary vectors spanning the tangent space (i.e. tangent and binormal vectors), for some α and β. If the light travels from x_i on one side of the boundary to x_o on the other, then the light path should extremize the Lagrangian

$$\mathcal{L}[\gamma] = \|y(\alpha,\beta) - x_i\|_{g^{(i)}} + \|y(\alpha,\beta) - x_o\|_{g^{(o)}} . \tag{19}$$

Taking derivatives in u and v,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \alpha} &= \frac{\langle u, y(\alpha,\beta) - x_i\rangle_{g^{(i)}}}{\|y(\alpha,\beta) - x_i\|_{g^{(i)}}} + \frac{\langle u, y(\alpha,\beta) - x_o\rangle_{g^{(i)}}}{\|y(\alpha,\beta) - x_o\|_{g^{(o)}}} = \frac{\langle u, \omega_i\rangle_{g^{(i)}}}{\|\omega_i\|_{g^{(i)}}} - \frac{\langle u, \omega_o\rangle_{g^{(o)}}}{\|\omega_o\|_{g^{(o)}}} = 0 \,, \\ \frac{\partial \mathcal{L}}{\partial \beta} &= \frac{\langle v, y(\alpha,\beta) - x_i\rangle_{g^{(i)}}}{\|y(\alpha,\beta) - x_i\|_{g^{(i)}}} + \frac{\langle v, y(\alpha,\beta) - x_o\rangle_{g^{(i)}}}{\|y(\alpha,\beta) - x_o\|_{g^{(o)}}} = \frac{\langle v, \omega_i\rangle_{g^{(i)}}}{\|\omega_i\|_{g^{(i)}}} - \frac{\langle v, \omega_o\rangle_{g^{(o)}}}{\|\omega_o\|_{g^{(o)}}} = 0 \,, \end{aligned} \tag{20}$$

where ω_i and ω_o are the ingoing and outgoing directions respectively. Therefore, given ω_i, we should choose ω_o to satisfy

$$\begin{aligned} \frac{\langle u, \omega_o\rangle_{g^{(o)}}}{\|\omega_o\|_{g^{(o)}}} &= \frac{\langle u, \omega_i\rangle_{g^{(i)}}}{\|\omega_i\|_{g^{(i)}}} \,, \\ \frac{\langle v, \omega_o\rangle_{g^{(o)}}}{\|\omega_o\|_{g^{(o)}}} &= \frac{\langle v, \omega_i\rangle_{g^{(i)}}}{\|\omega_i\|_{g^{(i)}}} \,. \end{aligned} \tag{21}$$

Note that, in the case where $g^{(i)}_{\mu\nu} = \eta_i^2 \delta_{\mu\nu}$ and $g^{(o)}_{\mu\nu} = \eta_o^2 \delta_{\mu\nu}$, the above equations reduce to

$$\eta_o \cos\theta_o = \eta_i \cos\theta_i \,, \tag{22}$$

the familiar Snell's Law. For the law of reflection, one repeats this procedure to obtain

$$\begin{aligned} \frac{\langle u, \omega_o\rangle_g}{\|\omega_o\|_g} &= -\frac{\langle u, \omega_i\rangle_g}{\|\omega_i\|_g} \,, \\ \frac{\langle v, \omega_o\rangle_g}{\|\omega_o\|_g} &= -\frac{\langle v, \omega_i\rangle_g}{\|\omega_i\|_g} \,. \end{aligned} \tag{23}$$

However, given ω_i and $g^{(i)}$ and $g^{(o)}$, it remains to actually solve for the refraction and reflection directions ω_o.

Computing Refraction and Reflection Vectors

To propose a numerical solution to solving the system (21), we first rewrite it the above using linear algebra notation. Let G_o and G_i be the 3x3 matrix representations of the metrics g_o and g_i. Note that G_o and G_i are SPD since g_o and g_i are metrics. We also choose u and v to be in the direction of ω_i.

Now, the system (21) can be written as

$$\begin{aligned} \frac{u^T G_o \omega_o}{\sqrt{\omega_o^T G_o \omega_o}} &= \frac{u^T G_i \omega_i}{\sqrt{\omega_i^T G_i \omega_i}} \,, \\ \frac{v^T G_o \omega_o}{\sqrt{\omega_o^T G_o \omega_o}} &= \frac{v^T G_i \omega_i}{\sqrt{\omega_i^T G_i \omega_i}} \,. \end{aligned} \tag{24}$$

Since the quantities on the right hand sides are known, we simply replace them by scalars γ and δ,

$$\begin{aligned} \frac{u^T G_o \omega_o}{\sqrt{\omega_o^T G_o \omega_o}} &= \gamma \,, \\ \frac{v^T G_o \omega_o}{\sqrt{\omega_o^T G_o \omega_o}} &= \delta \,. \end{aligned} \tag{25}$$

Now, we take the Cholesky factorization of $G_o = L^T L$. In comparison to the eigenvalue decomposition, this factorization is significantly cheaper and easier to compute. Now, when we make the substitution, we see that

$$\begin{aligned} \frac{(Lu)^T (L\omega_o)}{\sqrt{(L\omega_o)^T (L\omega_o)}} &= \gamma \,, \\ \frac{(Lv)^T (L\omega_o)}{\sqrt{(L\omega_o)^T (L\omega_o)}} &= \delta \,. \end{aligned} \tag{26}$$

Normalizing the vectors Lu and Lv, rewriting the normalized quantities as $\tilde{u} \equiv Lu/|Lu|$ and $\tilde{v} \equiv Lv/|Lv|$, and rewriting $\tilde{\omega}_o \equiv L\omega_o$ gives

$$\begin{aligned} \frac{\tilde{u}^T \tilde{\omega}_o}{|\tilde{\omega}_o|} &= \gamma' \,, \\ \frac{\tilde{v}^T \tilde{\omega}_o}{|\tilde{\omega}_o|} &= \delta' \,. \end{aligned} \tag{27}$$

We now transform all vectors by a rotation such that $\tilde{u}$ is rotated into the unit z vector and $\tilde{v}$ is rotated into the xz-plane. We call this orthogonal transformation Q. Then the above equations become

$$\begin{aligned} \cos\theta &= \gamma' \\ \cos\psi_{\omega v} &= \delta' \end{aligned} \tag{28}$$

Where θ is the inclination of transformed vector $Q\tilde{\omega}_o$ in spherical coordinates. And $\psi_{\omega v}$ is the angle $Q\tilde{\omega}_o$ makes with $Q\tilde{v}$. Furthermore, we can compute

$$\cos\psi_{uv} = \tilde{u}^T \tilde{v} \tag{29}$$

Then, the spherical law of cosines tells us that azimuth φ of $Q\tilde{\omega}_o$ is given by

$$\cos\varphi = \frac{\cos\psi_{\omega v} - \cos\theta\cos\psi_{uv}}{\sin\theta\sin\psi_{uv}} = \frac{\delta' - \gamma' \tilde{u}^T \tilde{v}}{\sqrt{1-\gamma'^2}\sqrt{1-(\tilde{u}^T\tilde{v})^2}} \tag{30}$$

Afterwards, there are two possibilities for the vector $QL\omega_o$,

$$QL\omega_o = (\sin\theta\cos\varphi, \pm\sin\theta\sin\varphi, \cos\theta) \tag{31}$$

Afterwards, we can recover the two possibilities for ω_o by applying Q^{-1} and then L^{-1}. We then take the vector which has positive dot product with the outgoing normal vector. It is possible that neither of the vectors has positive dot product, in which case, there is total internal reflection. One notes the same process can be used for computing reflection vectors in (23).

Example 1: Radial Transformations and An Imperfect Cloaking Device

To implement an example, we consider a simple class of metamaterial which transforms the radial distance of points from the origin. That is, we consider transformations of the form

$$\Phi(x) = \alpha(x)x \,. \tag{32}$$

That is the position $x = (x_1, x_2, x_3)$ is scaled by a factor α which depends on position. Note that Φ^{-1} also has the form of the above transformation. The Jacobian of this transformation is given by

$$D\Phi(x) = x \otimes \nabla\alpha(x) + \alpha(x)I \,, \tag{33}$$

where $\otimes$ denotes the outer product, and I is the 3x3 identity. Alternatively, these transformations can be written as sending $r \mapsto r'$ where $r = |x|$ and $\alpha(x) = r'/r$. Transformations with gaps in the coverage of r' mask regions of the image of Φ, thereby making them invisible. An example such transformation can be seen in figure (2).

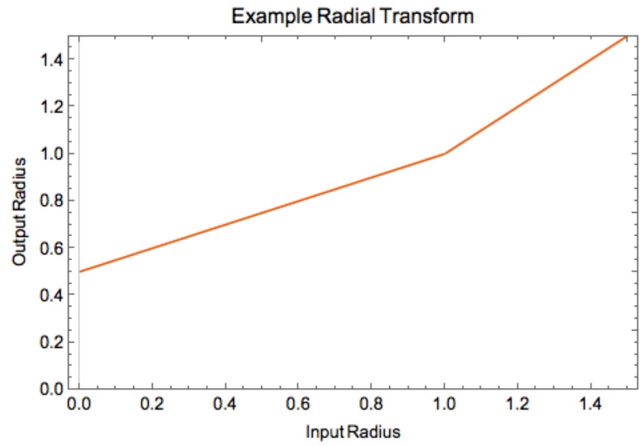

Figure 5: An example radial transformation with $R_1 = 0.5$ and $R_2 = 1$. This transformation generates a cavity of radius R_1 which light cannot enter. The kink at R_2 means that the resulting transformation does not generate a perfect invisibility device. One could generate a perfect invisibility device by enforcing that $r(0) = R_1$, $r(R_2) = R_2$ and $r'(R_2) = 1$.

We use this linear transformation instead of a transformation which does not have a kink at R_2 because this transformation is easier to implement and will produce an actual visual effect. Here $\alpha(x)$ and $\alpha^{-1}(x)$ are given by

$$\begin{aligned} \alpha(x) &= \frac{R_1}{|x|} + \frac{R_2 - R_1}{R_2}\,, \\ \alpha^{-1}(x) &= -\frac{R_1 R_2}{(R_2 - R_1)|x|} + \frac{1}{R_2 - R_1}\,, \end{aligned} \tag{34}$$

with corresponding gradients:

$$\begin{aligned} \nabla\alpha(x) &= -\frac{R_1 x}{|x|^3}\,, \\ \nabla\alpha^{-1}(x) &= \frac{R_1 R_2 x}{(R_2 - R_1)|x|^3}\,, \end{aligned} \tag{35}$$

The availability of all $\Phi, \Phi^{-1}, D\Phi, D\Phi^{-1}$ means that the coordinate transformed distance estimator algorithm is relatively straightforward to implement. To generate images, we use a simple signed distance estimator for a spherical shell with inner radius R_1 and outer radius R_2,

$$f(x) = \max(|x| - R_2, R_1 - |x|)\,. \tag{36}$$

The image generated by this technique can be seen in figure (7), with an image of the scene without the imperfect cloaking device seen in figure (6).

Example 2: Negative and Anisotropic Refraction

Another interesting feature of metamaterials is that they can generate negative refraction, and also refraction which is anisotropic. For this example, I use the transformation given by

$$\Phi^{-1}(x, y, z) = \begin{bmatrix} 0.75 & 0 & 0.1 \\ 0.1 & 0.85 & 0 \\ 0.7 & -0.1 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}\,. \tag{37}$$

The result of using this transformation can be seen in figure (8). Note that near the edges of the sphere, we can see total external reflection. This effect happens when the algorithm for computing refraction cannot find a suitable vector in the direction of the inward normal vector.

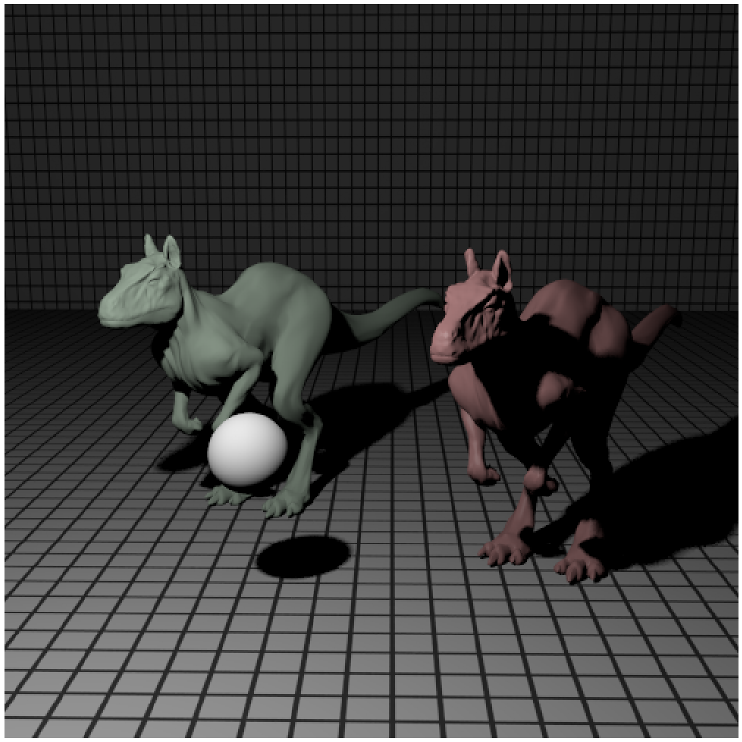

Figure 6: An example of our base scene with no imperfect cloaking device. The white sphere in front of the green Killeroo is obscured by an imperfect cloaking device in figure (7). Scene was rendered with direct integration.

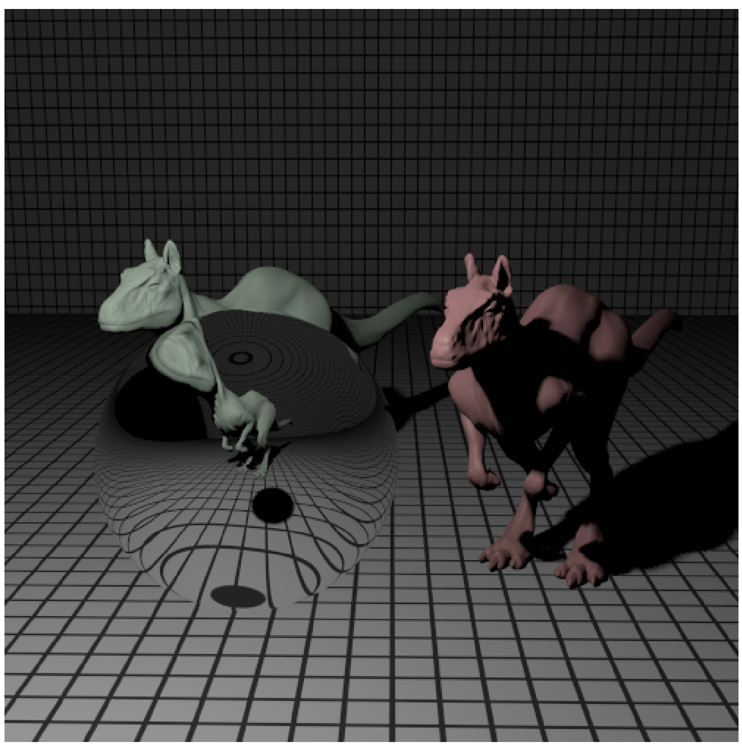

Figure 7: A copy of the above scene seen in figure (6), but with an imperfect cloaking device placed around the white sphere. Note that the cloaking device is perceptible because of the kink in the radial transformation from figure (5). However, the white sphere is not visible. Note that because the scene was rendered with only direct integration, the shadow of the white sphere is visible in the cloaking device. This is technically not correct, but it is a result of the integration method used and not the metamaterial algorithm.

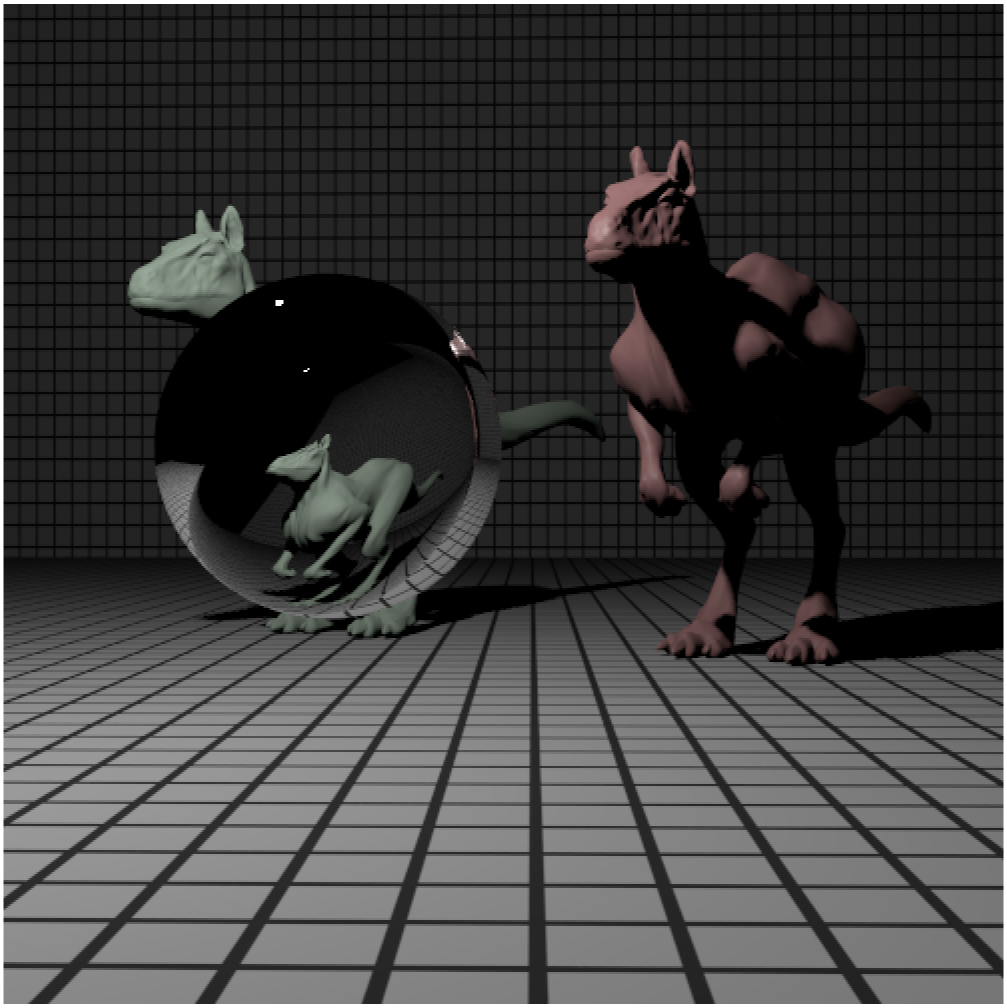

Figure 8: Negative anisotropic refraction generated by the transformation in equation (37). Note that near the edges of the sphere there is total external reflection.

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