More on Subdivision Surfaces and Smoothness

Lecture #16: Tuesday, 20 November 1997
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Last time, we talked about various subdivision schemes for two-dimensional surfaces. We will now clear up the mystery of where the coefficients in those methods came from, by examining how those coefficients influence the smoothness of subdivision surfaces.

1 What Is a Smooth Surface?

We shall start with an exact definition of a smooth surface. Informally, we would like to express the concept that a smooth surface is "a smooth two-dimensional subset of space." In other words, we need to define some subset of (three-dimensional) space that has a non-degenerate two-variable parameterization that is continuous and differentiable. To make this concept rigorous requires a somewhat complicated definition:

Definition 1. *A* ***simple smooth surface*** *in three-dimensional space is defined as a subset A of space, such that the intersection of A and a neighborhood of a point X in A has a regular parameterization p from the unit disc. Thus, if $A \subset \mathbf{R}^3$, B_X is the neighborhood of some $X \in A$, and $D \subset \mathbf{R}^2$ is the unit disc, A is a simple smooth surface if and only if $D \xrightarrow{p} A \bigcap B_X$. A* ***regular parameterization*** *p is one that is continuously differentiable, one-to-one, and has a Jacobi matrix of maximum rank.*

This definition corresponds directly to the intuition above. We pick a point X on our candidate surface A and examine its immediate neighborhood. This will be some region of space that intersects A. We claim that we have a simple smooth surface only if that region of intersection (which will be some patch of A) has a continuously differentiable one-to-one mapping onto a disc.

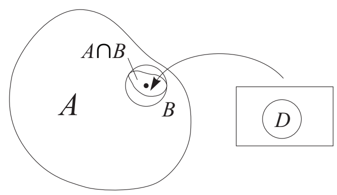

Figure 1: Definition of a simple smooth surface.

The condition that the Jacobi matrix of p have maximum rank is necessary to make sure we have no degeneracies (i.e. that we really do have a surface, not a curve or point). If $p = (p_1, p_2, p_3)$ and the disc is parameterized by x_1 and x_2, the condition is that the matrix

$$\begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} \\ \frac{\partial p_3}{\partial x_1} & \frac{\partial p_3}{\partial x_2} \end{pmatrix}$$

have maximal rank, i.e. two.

Although the above definition is certainly adequate for some applications, it is convenient to have a slightly different definition. The problem arises when we consider surfaces that intersect themselves – no self-intersecting surface can be a simple smooth surface. This becomes obvious when we consider a point on the line of intersection, since its neighborhood in three-dimensional space consists of pieces of *two* smooth surfaces.

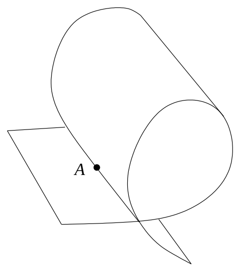

Figure 2: Self-intersecting surface - it is not simply smooth at A.

We would prefer to have a definition of smoothness that is more local, and independent of global properties such as self-intersection. If we perform some topological deformation on a smooth surface, we would like the surface to continue to be classified as smooth, even though the deformation might cause it to intersect itself. Therefore, we introduce a modified definition:

Definition 2. *A surface $A \subset \mathbf{R}^3$ is **smooth** (i.e. C^1) if and only if*

- *There exists a mapping f from a parametric domain $M \subset \mathbf{R}^2$ onto A.*
- *For every point $X \in M$ there exists a regular parameterization of the unit disc onto $f(U)$, where U is the neighborhood in M of X.*

We add an extra mapping f into the definition to provide a way of defining neighborhoods on the surface itself.

This definition is convenient when working with subdivision surfaces, since it can be shown that the mapping f always exists as we subdivide. In particular, consider a triangular mesh in the vicinity of an (ordinary or extraordinary) vertex of degree k. In the original mesh, we just let f be the mapping from a regular k-gon to a portion of

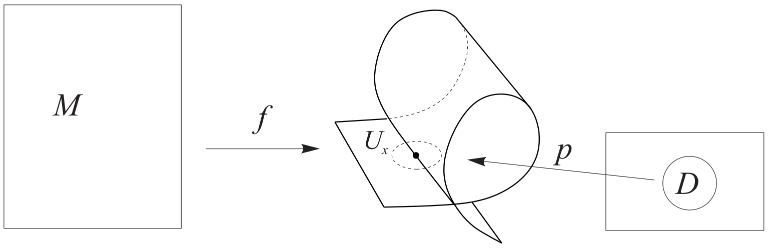

Figure 3: Definition of a C^1 (i.e. smooth) surface.

the mesh. As we subdivide the mesh, we subdivide the k-gon using the midpoint rule. We see that each added vertex in M corresponds exactly to a vertex added to the mesh. Note that although the mapping f defined in this way need not be smooth, this is not a requirement for the definition of a smooth surface – the only requirements are existence and smoothness p.

There is yet another definition of smoothness encountered from time to time. This is the notion of geometric smoothness:

Definition 3. *A surface A is* ***geometrically smooth*** *at $X \in A$ if and only if surface normals are defined in a neighborhood around X and there exists a limit of normals at X.*

This is a useful definition, since it is easier to prove surfaces geometrically smooth (all that is required is to show the existence of a limit), and also since the definition is very intuitive (it captures the notion that a surface is smooth if there exists a tangent plane). Geometric smoothness, however, is weaker than C^1 smoothness. Geometric smoothness at X implies the existence of a parameterization from the unit disc that is regular everywhere except at X. This contrasts with the regular definition of smoothness, which requires the parameterization to be regular everywhere.

As a simple example of a surface that is geometrically smooth but not C^1, consider the shape in Figure 4. Points in the vicinity of the central point are "wrapped around twice", so while there exists a tangent plane at that point, the surface does not "locally look like a plane". Formally, there does not exist a regular parameterization from the unit disc onto the neighborhood of the center point, even though that parameterization is regular in the neighborhood of that point.

From the previous example, we see how the definition of geometric smoothness must be strengthened to become C^1:

Lemma 4. *If a surface is geometrically smooth at a point and the projection of the surface onto the tangent plane at that point is one-to-one, the surface is C^1.*

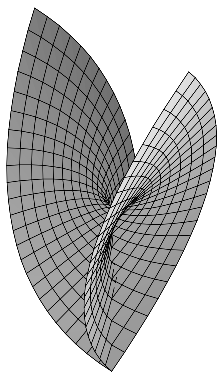

Figure 4: Example of a surface that is geometrically smooth but not C^1.

2 Smoothness of Subdivision Surfaces

We now turn to an examination of the smoothness properties of a subdivision surface. We will first examine the Loop scheme, and show how the eigenvalues of the subdivision matrix determine how smooth the limiting surface will be.

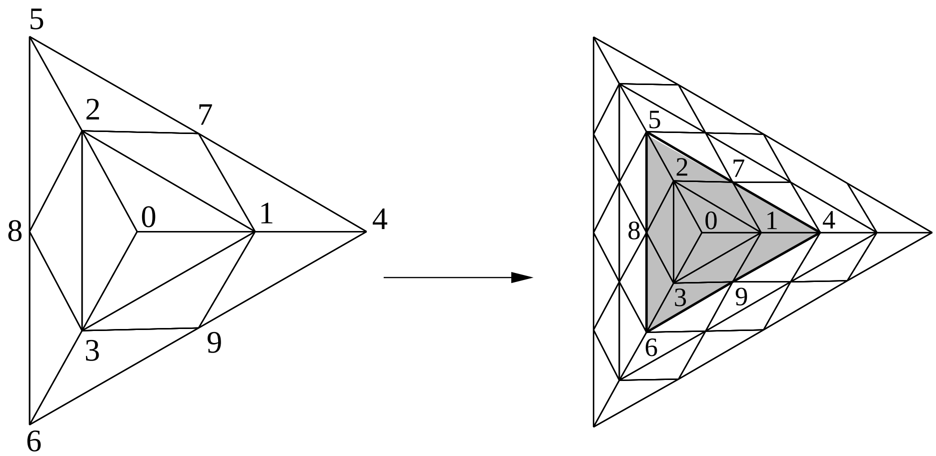

Figure 5: The Loop subdivision scheme near a vertex of degree 3. Note that $3(3)+1 = 10$ points in two rings are required.

Recall that in the Loop scheme only two "rings" of triangles around a vertex that define the surface patch surounding the vertex. So, if we let c_i^k be the coordinate of the point c_i at subdivision iteration k, we can represent the effect of subdivision on a point by

$$\begin{pmatrix} c_0^{k+1} \\ \vdots \\ c_{3n}^{k+1} \end{pmatrix} = \mathrm{S} \begin{pmatrix} c_0^{k} \\ \vdots \\ c_{3n}^{k} \end{pmatrix}$$

In this equation, n is the degree of the vertex, so two rings of triangles around it require $3n+1$ points. S is the local portion of the subdivision matrix. Note that the coordinates c_i^k are themselves vectors in three-space.

We can now rewrite each of the coordinate vectors in terms of the eigenvectors of the matrix S. Thus,

$$c^0 = \begin{pmatrix} c_0^{0} \\ \vdots \\ c_{3n}^{0} \end{pmatrix} = \sum_i a_i x_i,$$

and

$$c^k = (\mathrm{S})^k c^0 = \sum_i (\lambda_i)^k a_i x_i,$$

where the x_i are the eigenvectors of S, and the λ_i are the corresponding eigenvalues, arranged in nonincreasing order. As stated in an earlier lecture, we require λ_0 to be 1 for all subdivision schemes, in order to guarantee invariance with respect to translations and rotations. Furthermore, all stable, converging subdivision schemes will have all the remaining λ_i less than 1.

It is clear that as we subdivide, the behavior of c^k, which determines the behavior of the surface in the immediate vicinity of our point of interest, will depend only on the largest eigenvalues of S. In the limit as $k \to \infty$, we care only about λ_0 and λ_1 for curves; for surfaces, we also care about λ_2.

To proceed with the derivation, we will assume for simplicity that $\lambda_1 \geq \lambda_2 > \lambda_3$. Furthermore, we let $a_0 = 0$ – recall that this corresponds to choosing the origin of our coordinate system in the limit position of the vertex of interest. Then we can write

$$\frac{c^k}{(\lambda_1)^k} = a_1 x_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k a_2 x_2 + \cdots,$$

where the higher-order terms disappear in the limit.

We now have two cases to consider – we will first consider what happens if $\lambda_1 = \lambda_2$. In this case

$$\lim_{k\to\infty} \frac{c^k}{(\lambda_1)^k} = a_1 x_1 + a_2 x_2.$$

This means that, up to a scaling by $(\lambda_1)^k$, the control points approach a fixed configuration. This configuration is determined by x_1 and x_2, which depend only on the subdivision scheme chosen, and on a_1 and a_2, which are determined by the initial conditions. Note, however, that given any non-degenerate initial conditions, the result will be the same up to an affine transform. Therefore, in some sense, all surfaces subdivided using a given rule "look the same" in the limit, and the limiting configuration depends on the eigenvectors x_1 and x_2 of the subdivision matrix.

Now recall that a_1 and a_2 are just vectors in $\mathbf{R}^3$, while x_1 and x_2 are $3n+1$-component vectors, one component per control point. Therefore, we see that the expression $a_1 x_1 + a_2 x_2$ must represent a set of control points that all lie in one plane. So in the limit a subdivision rule with $\lambda_0 > \lambda_1 = \lambda_2 > \lambda_3$ leads to smooth surfaces.

Now consider what happens when $\lambda_1 > \lambda_2$. In this case, the control points still approach a plane, but they "stretch" in one direction more than another. Therefore we still get smooth surfaces. Note that we must be careful to consider both λ_1 and λ_2, and not come to the erroneous conclusion that the control points might approach a line and therefore form a crease.

In the next lecture we will formulate a sufficient condition for smoothness.