Analysis of Subdivision Surfaces, Multiresolution Surfaces

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In which Denis discusses the smoothness of subdivision surfaces, introduces multiresolution surfaces and brings subdivision to its conclusion.

1 The Smoothness of Subdivision Surfaces

1.1 Introduction

The condition for smoothness of subdivision surfaces can be stated very simply. We could just start by explaining everything at the beginning.

Reif's sufficient condition for smoothness : Suppose the eigenvectors of a subdivision matrix form a basis, the largest eigenvalues three eigenvalues are real and satisfy

$$\lambda_0 = 1 > \lambda_1 \geq \lambda_2 > |\lambda_3|$$

. If the characteristic map is regular and one-to-one, then almost all surfaces generated by subdivision are smooth (C^1-continuous.)

We need to define several concepts mentioned in this criterion.

1.2 Subdivision surfaces as mappings

We will discuss subdivision on triangular meshes; constructions for schemes defined on quadrilateral meshes are similar. Suppose the initial control mesh for the surface has one extraordinary vertex of degree n and the rest of the vertices are regular (degree 6.) If we are interested only in local properties of subdivision surfaces, in fact this is the only kind of mesh we have to consider — any initial mesh after sufficient number of subdivision steps locally is identical to the mesh with one or none extraordinary vertices.

Consider a triangulation of the plane $\mathbf{R}^2$ which has similar structure: one vertex of degree n in the center, and all other vertices of degree 6. We can establish a one-to-one correspondence between the vertices of the triangulation of $\mathbf{R}^2$ and the vertices of the mesh $\mathbf{R}^3$; this defines a mapping from the set of vertices of the triangulation, which is a subset of the points of the plane, to the control vertices of the surface.

Suppose each time we apply the subdivision rules to compute the finer control mesh, we also apply midpoint subdivision to the triangulation of the plane. This means that we leave the old vertices where they are, and insert new vertices in the middle of each edge of the triangulation. Note that each control vertex that we insert in the mesh in $\mathbf{R}^3$ corresponds to a new vertex of the midpoint-subdivided triangulation.

As we repeat this process, we get a mapping from a denser and denser subset of the plane to the vertices of the finer and finer control mesh. In the limit we get a map from an everywhere dense set on the plane to the surface, which can be extended by continuity to the whole plane. As a result, we get a natural parameterization of the subdivision surface over the plane: $f : \mathbf{R}^2 \to \mathbf{R}^3$.

To analyze smoothness, we need to consider arbitrarily small piece of the surface around the extraordinary point. In particular, we can take the part of the surface defined over a n-gon U that consists out of triangles adjacent to the extraordinary vertex: $f : U \to \mathbf{R}^3$. Note that the n-gon can be chosen on any level of subdivision; let U^j be the n-gon after j subdivision steps.

We can construct the map f for $n = 6$; in this case, initially we have a regular triangulation on the plane. It is easy to show that for splines–based subdivision the mapping f in this case will be just the standard piecewise-polynomial mapping defining the spline.

In general, subdivision schemes are typically constructed in such way that the mapping f that we get for a regular initial mesh (with all vertices of degree 6) is C^1-continuous.

An important observation about the mapping f is that after sufficient number of subdivision steps in a neighborhood of any point inside the triangles of the initial triangulation of the plane, the map coincides with the map generated from a suitably chosen regular mesh (this is not true on the boundaries of triangles!)

In particular, f is C^1-continuous inside each triangle of the n-gon U^0 and we can compute the derivatives. To ensure smoothness of the surface, f should have Jacobi matrix of maximal rank.

It turns out that we need not consider all possible maps f to establish smoothness of subdivision. In most cases it is sufficient to consider a single map called the *characteristic map* which is completely defined by the subdivision scheme. To define this map, we need to discuss eigenvalues and eigenvectors of the subdivision matrix.

1.3 Characteristic Map

The part of the subdivision surface on U is completely defined by a set of the control vertices near the extraordinary vertex v. For example, for Loop's scheme the surface is defined by two rings of control vertices around v. For a vertex of degree n these rings contain $3n + 1$ vertices, including the extraordinary vertex itself. Let $c_i = (x_i, y_i)$ be the control vertices for the two rings, forming a vector $\mathbf{c}$ of length $3n + 1$:

$$c_i = (x_i, y_i) \tag{1}$$

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{3n} \end{bmatrix} \tag{2}$$

$\mathbf{c}^0$ is the initial vector of control vertices and $\mathbf{c}^j$ is the vector after j subdivisions. If we define $\mathbf{S}$ to be the subdivision matrix, then the act of subdividing can be represented as repeated multiplication of $\mathbf{c}$ by $\mathbf{S}$, exactly as in the case of the curves.

$$\mathbf{c}^{j+1} = \mathbf{S}\mathbf{c}^j \tag{3}$$

$$\mathbf{c}^{j+1} = \mathbf{S}^{j+1}\mathbf{c}^0 \tag{4}$$

After each subdivision the $3n+1$ control vertices defining the surface on the n-gon U^j are becoming closer and closer to the central vertex. One more remarkable thing happens: if we scale the control vertices $\mathbf{c}^j$, we will see that the configuration of the control vertices converges to a limit. To describe this more precisely, we have to look at the eigenvalues and eigenvectors of the subdivision matrix.

Suppose λ_i are the eigenvalues of $\mathbf{S}$, and they are numbered in nonincreasing order: $|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_{3n}|$. For simplicity we assume that there is a basis of eigenvectors of $\mathbf{S}$. λ_0, as usual, must be 1. For convergence we need $\lambda_1 < 1$. In addition, we assume that λ_1, λ_2 are real and $|\lambda_3| < \lambda_2$.

Then similar to the one-dimensional case that we have considered before,

$$\mathbf{c}^{j+1} \cong \lambda_1^n a_1 x_1 + \lambda_2^n a_2 x_2. \tag{5}$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{S}$, x_1 and x_2 are corresponding eigenvectors, and a_1 and a_2 are vectors in $\mathbf{R}^3$ defining the tangent plane.

We have used the usual trick of moving the origin of our coordinate system to the central point of the subdivision, so that a_0 is 0, and there are no terms containing λ_0. In this expression only a_1 and a_2 depend on the initial control vertices; the eigenvectors x_1 and x_2 depend only on the choice of the subdivision scheme.

Now we are ready to define the *characteristic map.* Note that when we described subdivision as a function from the plane to $\mathbf{R}^3$, we could have use control vertices not from $\mathbf{R}^3$ but from $\mathbf{R}^2$; clearly, subdivision rules can be applied in the plane rather then in space. Then in the limit we obtain a map from the plane into the plane. The characteristic map is the map of this type.

As we have seen, the configuration of control points near an extraordinary vertex approaches $a_1x_1 + a_2x_2$, up to a scaling transformation. This means that the part of the surface defined on the n-gon U^j as $j \to \infty$ approaches the surface defined by the vector of control points $a_1x_1 + a_2x_2$. Up to an affine transform which does not affect smoothness, in the limit all surfaces generated by subdivision are identical to the surface generated by the control vertices c_i, such that $c_i = ([x_1]_i, [x_2]_i, 0)$, that is, the first two coordinates are corresponding coordinates of the eigenvectors x_1 and x_2. The control vertices c_i are obtained from $a_1[x_1]_i + a_2[x_2]_i$ applying the affine transform taking a_1 to e_1 and a_2 to e_2, where e_1, e_2, e_3 are the basis vectors. By throwing away the last zero coordinate, we obtain a vector of control points in $\mathbf{R}^2$ completely defined by the eigenvectors of $\mathbf{S}$. This is a map from the n-gon U^0 to $\mathbf{R}^2$.

As we have discussed, inside each triangle of the n-gon U^0 the map is C^1. Moreover, the map has one-sided derivatives on the boundaries of the triangles, except at the extraordinary vertex, so we can define one-sided Jacobians on the boundaries of triangles too. Moreover, it is possible to show that the Jacobian actually has to be continuous away from the extraordinary vertex despite the fact that the derivatives may be not continuous. We will say that the characteristic map is *regular* if its Jacobian is not zero anywhere away from the extraordinary vertex (including the boundaries!)

Now we have all of the tools that we need to understand the condition given at the beginning:

Reif's sufficient condition for smoothness : Suppose the eigenvectors of a subdivision matrix form a basis, the largest eigenvalues three eigenvalues are real and satisfy

$$\lambda_0 = 1 > \lambda_1 \geq \lambda_2 > |\lambda_3|$$

. If the characteristic map is regular and one-to-one, then almost all surfaces generated by subdivision are smooth (C^1-continuous.)

2 Multiresolution Surfaces based on Subdivision

2.1 Problems With Subdivision

Subdivision is great, but there are some problems with it. One problem is that it's often easier to generate or manipulate a model with different levels of detail. For example, when manipulating a model of an armadillo, it's much easier to move an ear as a single object than to move each point that makes up the ear.

One solution for this problem (the one we will be considering) is multiresolution surfaces based on subdivision, We discuss how to use multiresolution surfaces, but will discuss how to generate these surfaces from arbitrary meshes. That is a separate and complicated topic.

2.2 Wavelet-Like Multiresolution Surfaces

2.2.1 Wavelets *A Blast From the Past*

As you recall, wavelets separated low-frequency and high-frequency components of an image. Another way of looking at this is that wavelets generate a signal of averages, and a signal of details.

We will now apply the same strategy to subdivision surfaces.

2.2.2 Comparison of Wavelets and Subdivision

The following figure is a diagram of a wavelet filter. The boxed section is analogous to subdivision, adding more points.

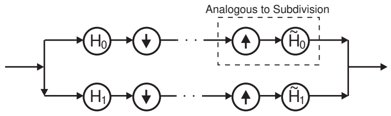

Figure 1: A wavelet filter bank.

If H_0 is known, it is typically possible to choose H_1, in such a way that they form a filter pair.

2.2.3 Another Strategy

If we are doing a hierarchical wavelet transform and we do not subsample the detail part of a signal, the resulting signal is two times larger than if we has subsampled. However, if we skip the subsampling in 2D, the resulting signal is only $\frac{4}{3}$ times larger.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 \tag{6}$$

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{4}{3} \tag{7}$$

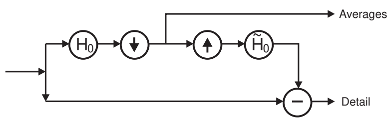

Figure 2: A filter bank that doesn't subsample the detail.

2.3 Performance of Multiresolution Surfaces

With full detail, the computer takes a long time to manipulate the model in the figure. However, by taking out levels of detail and then manipulating the model, everything becomes much faster.

Also, it is much easier to perform gross manipulations of the image when it has coarse detail. For example, if we want to move the ear, we don't have to move every control point, we just have to go to lower detail, move a few points, and then go back to high detail. The details follow the changes made at the coarse level.

Model manipulation is therefore improved in three different ways by using multiresolution surfaces.

1. It is faster because there is much less load on the computer.

2. It is faster because the user needs to manipulate fewer control points.

3. It is more intuitive, because the user can manipulate large sections of the image easily. For example, is is much more intuitive to "move the ear" than it is to "move all of the control points that comprise the ear".

See Figure 3, Figure 4, and Figure 5 for examples.

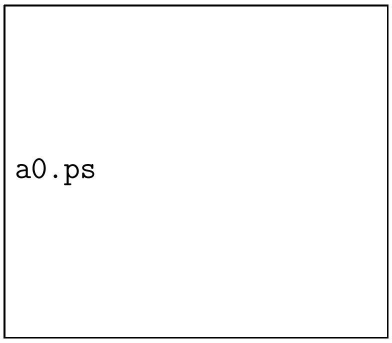

Figure 3: Armadillo Man

3 Improvements of Multiresolution Surfaces based on Subdivision

3.1 Dynamic Subdivision

Another problem with subdivision is that it sometimes gives too much detail. We do not always want the same resolution everywhere; we want more resolution only where

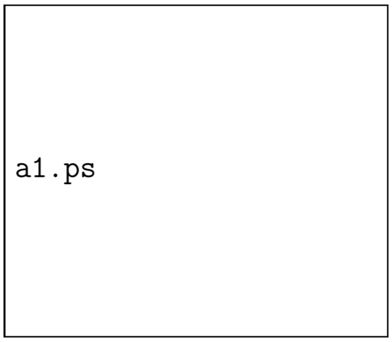

Figure 4: Armadillo Man at coarse resolution, with his ear moved.

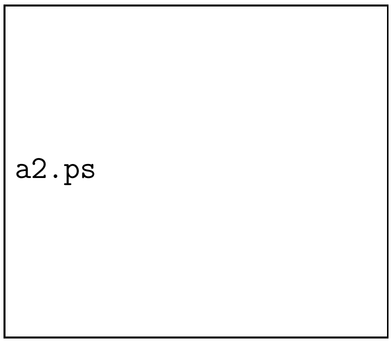

Figure 5: Armadillo Man at high resolution, with his ear moved.

we want a finer representation. In Figure 6, Figure 7, and Figure 8, more resolution is needed where the surface changes rapidly than where the surface remains fairly constant.

This problem can be solved by adaptively subdividing the surface. A triangle will only be subdivided if more subdivision is necessary. A flat plane, for example, should not be subdivided and given more detail, even if an adjacent jagged peak is being subdivided and given more detail.

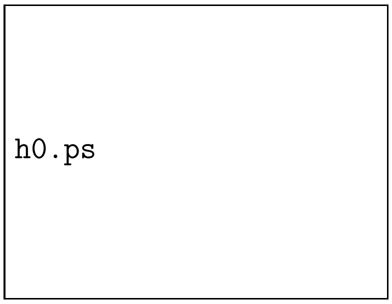

Figure 6: A fairly smooth model.

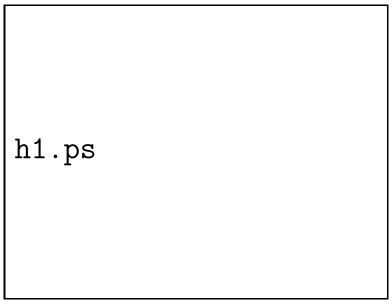

Figure 7: The same model with a slight bump.

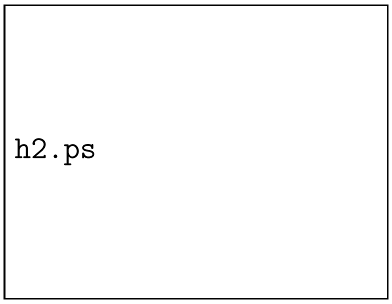

Figure 8: The same model with a large bump.

3.2 Adding Details Correctly

If a coarse version of an image is modified, care must be taken when adding back the details. If they are added blindly, the results may not be what is intended, as in the picture below.

A better strategy is to remember the normal of the surface where a detail is located, and to add the details relative to this normal.

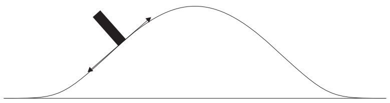

Figure 9: A figure with some detail.

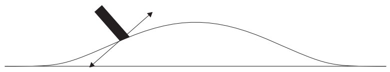

Figure 10: The figure was changed and the detail followed.

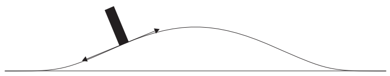

Figure 11: The detail was changed to reflect the new normal.