It is by logic we prove, it is by intuition that we invent.
— Henri Poincaré(1854–1912)

6 Homology

In Lecture 3, we learned about a combinatorial method for representing spaces. In Lecture 4, we studied groups and equivalence relations implied by their normal subgroups. In this lecture, we look at a combinatorial and computable functor called *homology* that gives us a finite description of the topology of a space. Homology groups may be regarded as an algebraization of the first layer of geometry in cell structures: how cells of dimension n attach to cells of dimension $n-1$ [1]. Mathematically, the homology groups have a less transparent definition than the fundamental group, and require a lot of machinery to be set up before any calculations. We focus on a weaker form of homology, simplicial homology, that both satisfies our need for a combinatorial functor, and obviates the need for this machinery. Simplicial homology is defined only for simplicial complexes, the spaces we are interested in. Like the Euler characteristic, however, homology is an invariant of the underlying space of the complex. Indeed, the invariance of the Euler characteristic is often derived from the invariance of homology.

Homology groups, unlike the fundamental group, are abelian. In fact, the first homology group is precisely the abelianization of the fundamental group. We pay a price for the generality and computability of homology groups: homology has less differentiating power than homotopy. Once again, however, homology respects homotopy classes, and therefore, classes of homeomorphic spaces.

6.1 Chains and Cycles

To define homology groups, we need simplicial analogs of paths and loops. Let K be a simplicial complex. Recall oriented simplices from Lecture 3. We create the chain group of oriented simplices on the complex.

Definition 6.1 (chain group) The *kth chain group* of a simplicial complex K is $\langle \mathsf{C}_k(K), + \rangle$, the free abelian group on the oriented k-simplices, where $[\sigma] = -[\tau]$ if $\sigma = \tau$ and σ and τ have different orientations. An element of $\mathsf{C}_k(K)$ is a k-*chain*, $\sum_q n_q[\sigma_q], n_q \in \mathbb{Z}, \sigma_q \in K$.

We often omit the complex in the notation. A simplicial complex has a chain group in every dimension. As stated earlier, homology examines the connectivity between two immediate dimensions. To do so, we define a structure-relating map between chain groups.

Definition 6.2 (boundary homomorphism) Let K be a simplicial complex and $\sigma \in K$, $\sigma = [v_0, v_1, \ldots, v_k]$. The boundary homomorphism $\partial_k : \mathsf{C}_k(K) \to \mathsf{C}_{k-1}(K)$ is

$$\partial_k \sigma = \sum_i (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n], \qquad (1)$$

where $\hat{v}_i$ indicates that v_i is deleted from the sequence.

It is easy to check that ∂_k is well-defined, that is, ∂_k is the same for every ordering in the same orientation.

Example 6.1 (boundaries) Let us take the boundary of the simplices in Figure 1. .

- $\partial_1[a, b] = b - a.$
- $\partial_2[a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b].$
- $\partial_3[a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c].$

Note that the boundary operator orients the faces of an oriented simplex. In the case of the triangle, this orientation corresponds to walking around the triangle on the edges, according to the orientation of the triangle.

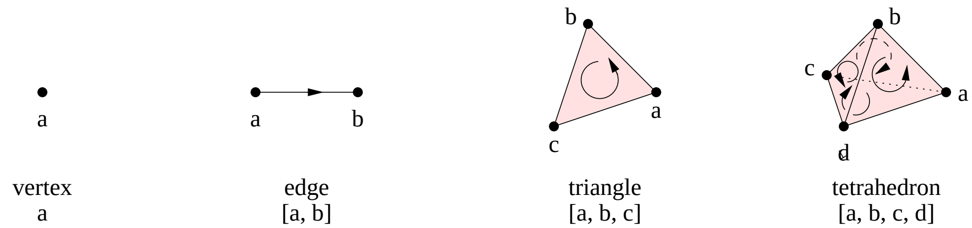


Figure 1. k-simplices, $0 \leq k \leq 3$. The orientation on the tetrahedron is shown on its faces.

If we take the boundary of the boundary of the triangle, we get:

$$\partial_1 \partial_2 [a, b, c] = [c] - [b] - [c] + [a] + [b] - [a] = 0. \tag{2}$$

This is intuitively correct: the boundary of a triangle is a cycle, and a cycle does not have a boundary. In fact, this intuition generalizes to all dimensions.

Theorem 6.1 $\partial_{k-1}\partial_k = 0$, *for all* k.

Proof: The proof is elementary.

$$\begin{aligned}
\partial_{k-1}\partial_k[v_0, v_1, \ldots, v_k] &= \partial_{k-1} \sum_i (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k] \\
&= \sum_{j<i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k] \\
&\quad + \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k] \\
&= 0,
\end{aligned}$$

as switching i and j in the second sum negates the first sum. □

The boundary operator connects the chain groups into a *chain complex* C_*:

$$\ldots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \ldots.$$

with $\partial_k \partial_{k+1} = 0$ for all k. For generality, we often define null boundary operators in dimensions where the domain or codomain of the boundary operator is empty, e.g. $\partial_0 \equiv 0$. A chain complex C_* should be viewed as a single object. Chain complexes are common in homology, but this particular chain complex is one of two we will see in our class.

The boundary operator also allows us to define subgroups of C_k: the group of *cycles* and the group of *boundaries*.

Definition 6.3 (cycle group, boundary group) The k*th cycle group* is

$$\begin{aligned}
\mathsf{Z}_k &= \ker \partial_k \\
&= \{c \in \mathsf{C}_k \mid \partial_k c = \emptyset\}.
\end{aligned}$$

A chain that is an element of Z_k is a k*-cycle*. The k*th boundary group* is

$$\begin{aligned}
\mathsf{B}_k &= \operatorname{im} \partial_{k+1} \\
&= \{c \in \mathsf{C}_k \mid \exists d \in \mathsf{C}_{k+1} : c = \partial_{k+1} d\}.
\end{aligned}$$

A chain that is an element of B_k is a k*-boundary*. We also call boundaries *bounding cycles* and cycles not in B_k *non-bounding cycles*.

Both subgroups are normal because our chain groups are abelian. The names match the names we had for loops in the fundamental group, but also extend the notions to other dimensions. Bounding cycles *bound* higher dimensional cycles, as otherwise they would not be in the image of the boundary homomorphism. We can think of them as "filled" cycles, as opposed to "empty" non-bounding cycles. The definitions of the subgroups, along with Theorem 6.1, imply that the subgroups are nested, $\mathsf{B}_k \subseteq \mathsf{Z}_k \subseteq \mathsf{C}_k$, as shown in Figure 2.

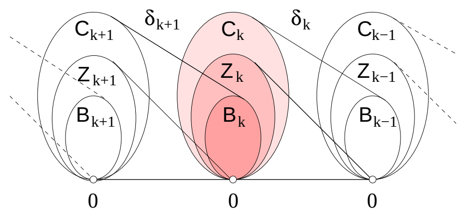


Figure 2. A chain complex with its internals: chain, cycle, and boundary groups, and their images under the boundary operators.

6.2 Simplicial Homology

Chains and cycles are simplicial analogs of the maps called paths and loops in the continuous domain. Following the construction of the fundamental group, we now need a simplicial version of a homotopy to form equivalent classes of cycles. Consider the sum of the non-bounding 1-cycle and a bounding 1-cycle in Figure 3. The two cycles z, b have

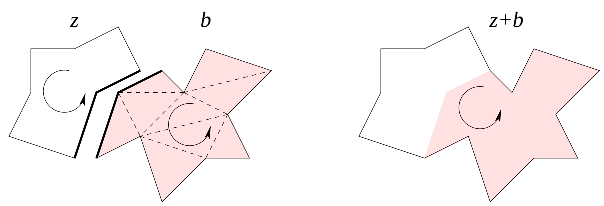


Figure 3. A non-bounding oriented 1-cycle $z \in \mathsf{Z}_k, z \notin \mathsf{B}_k$ is added to a oriented 1-boundary $b \in \mathsf{B}_k$. The resulting cycle $z + b$ is homotopic to z. The orientation on the cycles is induced by the arrows.

a shared boundary. The edges in the shared boundary appear twice in the sum $z + b$ with opposite signs, so they are eliminated. The resulting cycle $z + b$ is homotopic to z: we may slide the shared portion of the cycles smoothly across the triangles that b bounds. But such homotopies exist for any boundary $b \in \mathsf{B}_1$. Generalizing this argument to all dimensions, we look for equivalent classes of $z + \mathsf{B}_k$ for a k-cycle. But these are precisely the cosets of B_k in Z_k

As B_k is normal in Z_k, the cosets form a group under coset addition.

Definition 6.4 (homology group) The *kth homology group* is

$$\mathsf{H}_k = \mathsf{Z}_k / \mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}. \tag{3}$$

If $z_1 = z_2 + \mathsf{B}_k, z_1, z_2 \in \mathsf{Z}_k$, we say z_1 and z_2 are *homologous* and denote it with $z_1 \sim z_2$.

Homology groups are finitely generated abelian. Therefore, the fundamental theorem of finitely generated abelian groups from Lecture 4 applies. Homology groups describe spaces through their Betti numbers and the torsion subgroups.

Definition 6.5 (kth Betti Number) The *kth Betti number β_k of a simplicial complex K* is $\beta_k = \beta(\mathsf{H}_k)$, the rank of the free part of H_k.

We can show that $\beta_k = \operatorname{rank} \mathsf{H}_k = \operatorname{rank} \mathsf{Z}_k - \operatorname{rank} \mathsf{B}_k$. The description given by homology is finite, as a n-dimensional simplicial space has at most $n + 1$ nontrivial homology groups.

6.3 Understanding Homology

The description provided by homology groups may not be transparent at first. In this section, we look at a few examples to gain an intuitive understanding of what homology groups capture. Table 1 lists the homology groups of the basic 2-manifolds we first met in Lecture 2. As they are 2-manifolds, the highest non-trivial homology group for any of

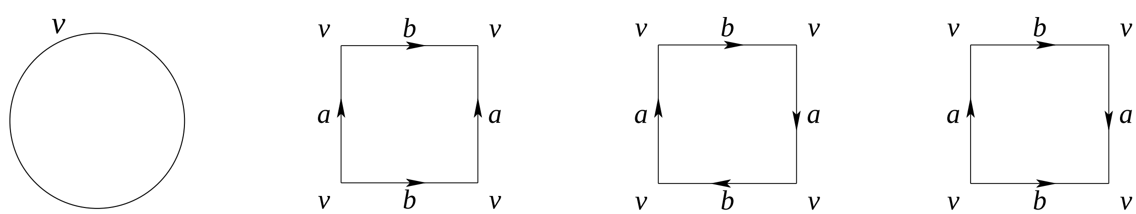
2-manifold	H_0	H_1	H_2
sphere	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$
torus	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$
projective plane	$\mathbb{Z}$	$\mathbb{Z}_2$	$\{0\}$
Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\{0\}$

Table 1. Homology of basic 2-manifolds.

them is H_2. *Torsion-free* spaces have homology that does not have a torsion subgroup, that is, terms that are finite cyclic groups Z_m. Most of the spaces we are interested are torsion-free. In fact, any space that is a subcomplex of $\mathbb{S}^3$ is torsion-free. We deal with $\mathbb{S}^3$ as it is compact. $\mathbb{R}^3$ is not compact and creates special cases that need to be handled in algorithms. To avoid these difficulties, we add a point at infinity and *compactify* $\mathbb{R}^3$ to get $\mathbb{S}^3$, the three-dimensional sphere. This construction mirrors that of the two-dimensional sphere in Lecture 2. Algorithmically, the *one point compactification of* $\mathbb{R}^3$ is easy, as we have a simplicial representation of space.

So what does homology capture? For torsion-free spaces in three-dimensions, the Betti numbers (the number of $\mathbb{Z}$ terms in the description) have intuitive meaning as a consequence of the *Alexander Duality*. β_0 measures the number of components of the complex. β_1 is the rank of a basis for the *tunnels*. As H_1 is free, it is a vector-space and β_1 is its rank. β_2 counts the number of *voids* in the complex. Tunnels and voids exist in the complement of the complex in $\mathbb{S}^3$. The distinction might seem tenuous, but this is merely because of our familiarity with the terms. For example, the complex *encloses* a void, and the void is the empty space *enclosed* by the complex.

Using this understanding, we may now examine Table 1. All four spaces have a single component, so $\mathsf{H}_0 = \mathbb{Z}$ and $\beta_0 = 1$. The sphere and the torus enclose a void, so $\mathsf{H}_2 = \mathbb{Z}$ and $\beta_2 = 1$. The non-orientable spaces, on the other hand, are one-sided and cannot enclose any voids, so they have trivial homology in dimension 2. To see what H_1 captures, we look at the diagrams for the 2-manifolds in Figure 4. We may, of course, triangulate these diagrams to



(a) Sphere $\mathbb{S}^2$ (b) Torus $\mathbb{T}^2$ (c) Projective plane $\mathbb{RP}^2$ (d) Klein bottle $\mathbb{K}^2$

Figure 4. Diagrams for basic 2-manifolds.

obtain abstract simplicial complexes for computing simplicial homology. For now, though, we assume that whatever curve we draw on these manifolds could be "snapped" to some triangulation of the diagrams. To understand 1-cycles and torsion, we need to pay close attention to the boundaries in the diagrams. Recall that a boundary is simply a cycle that *bounds*. In each diagram, we have a boundary, simply, the boundary of the diagram! The manner in which this boundary is labeled determines how the space is connected, and therefore the homology of the space.

It is clear that any simple closed curve drawn on the disk for the sphere is a boundary. Therefore, its homology is trivial in dimension one. The torus has two classes of non-bounding cycles. When we glue the edges marked 'a', edge 'b' becomes a non-bounding 1-cycle and forms a class with all 1-cycles that are homologous to it. We get a different class of cycles when we glue the edges marked 'b'. Each class has a generator, and each generator is free to generate as many *different* classes of homologous 1-cycles as it pleases. Therefore, the homology of a torus in dimension one is $\mathbb{Z} \times \mathbb{Z}$ and $\beta_1 = 2$.

There is a 1-boundary in the diagram, however: the boundary of the disk that we are gluing. Going around this 1-boundary, we get the description $aba^{-1}b^{-1}$. That is, the disk makes the cycle with this description a boundary. Equivalently, the disk adds the relation $aba^{-1}b^{-1} = 1$ to the presentation of the group. But this relation is simply stating that the group is abelian and we already knew that.

Continuing in this manner, we look at the boundary in the diagram for the projective plane. Going around, we get the description $abab$. If we let $c = ab$, the boundary is c^2 and the disk adds the relation $c^2 = 1$ to the group presentation.[1] In other words, we have a cycle c in our manifold that is non-bounding, but becomes bounding when we go around it twice. If we try to generate all the different cycles from this cycle, we just get two classes: the class of cycles homologous to c, and the class of boundaries. But any group with two elements is isomorphic to $\mathbb{Z}_2$, hence the description of H_1. You should convince yourself of the verity of the description of H_1 for the Klein bottle in a similar fashion.

6.4 Invariance

Like the Euler characteristic, we define homology using simplicial complexes. From the definition, it seems that homology is capturing extrinsic properties of our representation of a space. We are interested in intrinsic properties of the space, however. We hope that any two different simplicial complexes K and L with homeomorphic underlying spaces $|K| \approx |L|$ have the same homology, the homology of the space itself. Poincaré stated this hope in terms of "the principal conjecture" in 1904.

Conjecture 6.1 (Hauptvermutung) *Any two triangulations of a topological space have a common refinement.*

In other words, the two triangulations can be subdivided until they are the same. This conjecture, like Fermat's last lemma, is deceptively simple. Papakyriakopoulos verified the conjecture for polyhedra of dimension ≤ 2 in 1943 [7], and Moïse proved it for three-dimensional manifolds in 1953 [5]. Unfortunately, the conjecture is false in higher dimensions for general spaces. Milnor obtained a counterexample in 1961 for dimensions six and greater using Lens spaces [4]. Kirby and Siebenmann produced manifold counterexamples in 1969 [2]. The conjecture fails to show the invariance of homology [8].

To settle the question of topological invariance of homology, a more general theory was introduced, that of *singular homology*. This theory is defined using maps on general spaces, thereby eliminating the question of representation. Homology is axiomatized as a sequence of functors with specific properties. Much of the technical machinery required is for proving that singular homology satisfies the axioms of a homology theory, and that simplicial homology is equivalent to singular homology. Mathematically speaking, this machinery makes homology less transparent than the fundamental group. Algorithmically, however, simplicial homology is the ideal mechanism to compute topology.

6.5 The Euler-Poincaré Formula

Let's revisit the Euler characteristic now in our new setting. We may redefine the Euler characteristic over a chain complex.

Definition 6.6 (Euler characteristic) $\chi(\mathsf{C}_*) = \sum_i (-1)^i \operatorname{rank}(\mathsf{C}_i)$.

This definition is trivially equivalent to our previous one as the k-simplices are the generators of C_k and $\operatorname{rank}(\mathsf{C}_i) = |\{\sigma \in K \mid \dim \sigma = i\}|$ in that definition. We now denote the sequence of homology functors as H_*. Then, $\mathsf{H}_*(\mathsf{C}_*)$ is a chain complex:

$$\ldots \to \mathsf{H}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{H}_k \xrightarrow{\partial_k} \mathsf{H}_{k-1} \to \ldots.$$

The operators between the homology groups are induced by the boundary operators: we map a homology class to the class of the boundary of one of its members. According to the new definition, the Euler characteristic of our new chain is

$$\mathsf{H}_*(\mathsf{C}_*) = \sum_i (-1)^i \operatorname{rank}(\mathsf{H}_i) = \sum_i (-1)^i \beta_i$$

. Surprisingly, the homology functor preserves the Euler characteristic of a chain complex.

[1] We need this substitution as an artifact of using this diagram, which we are using for adding some form of uniformity to our treatment. The definition of the cross-cap in Conway's ZIP proof, however, is the one we need here.

Theorem 6.2 (Euler-Poincaré) $\chi(K) = \chi(\mathsf{C}_*) = \chi(\mathsf{H}_*(\mathsf{C}_*))$. *That is,* $\sum_i (-1)^i s_i = \sum_i (-1)^i \beta_i$, *where* $s_i = |\{\sigma \in K \mid \dim \sigma = i\}|$ *and* $\beta_i = \operatorname{rank} \mathsf{H}_i$.

The theorem derives the invariance of the Euler characteristic from the invariance of homology.

Example 6.2 We know that the Euler characteristic of a sphere is 2. The Euler-Poincaré relation tells us where this 2 comes from. According to the relation, $\chi(\mathbb{S}^2) = \beta_0 - \beta_1 + \beta_2$. We have $\beta_0 = 1$, as the sphere has one component, $\beta_1 = 0$ as all 1-cycles are contractible, and $\beta_2 = 1$ as the sphere encloses a single void. Similarly, $\chi(\mathbb{T}^2) = 0$, as it has the same Betti numbers as the sphere, except that $\beta_1 = 2$.

Acknowledgments

The actual content of this lecture comes from Munkres [6] and Hatcher [1]. As always, the pedagogical constructions are mine. Massey includes a description of the Alexander Duality [3]. The other citations are referenced within the text.

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