COMPUTING HOMOLOGY:

$$\left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_{l_k} & \\ \hline & 0 & & 0 \\ & & & \end{array}\right]$$

CS 468 – Lecture 7

11/6/2

TIDBITS

- Lecture 8 is on Tuesday, November 12
- Email me about projects!
- Projects will be November 27th and December 4th.
- November 20th?
- Triangulation of example

(LAST TIME)

EULER-POINCARÉ

- chain complex C_*:

$$\ldots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \ldots$$

- Homology functors H_*
- $\mathsf{H}_*(\mathsf{C}_*)$ is a chain complex:

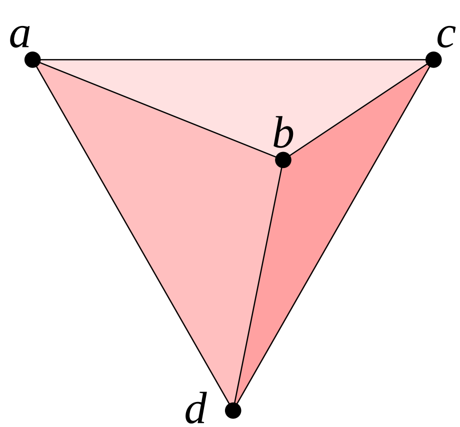
$$\ldots \to \mathsf{H}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{H}_k \xrightarrow{\partial_k} \mathsf{H}_{k-1} \to \ldots$$

- What is its Euler characteristic?
- (Theorem) $\chi(K) = \chi(\mathsf{C}_*) = \chi(\mathsf{H}_*(\mathsf{C}_*))$.
- $\sum_i (-1)^i s_i = \sum_i (-1)^i \operatorname{rank}(\mathsf{H}_i) = \sum_i (-1)^i \beta_i$

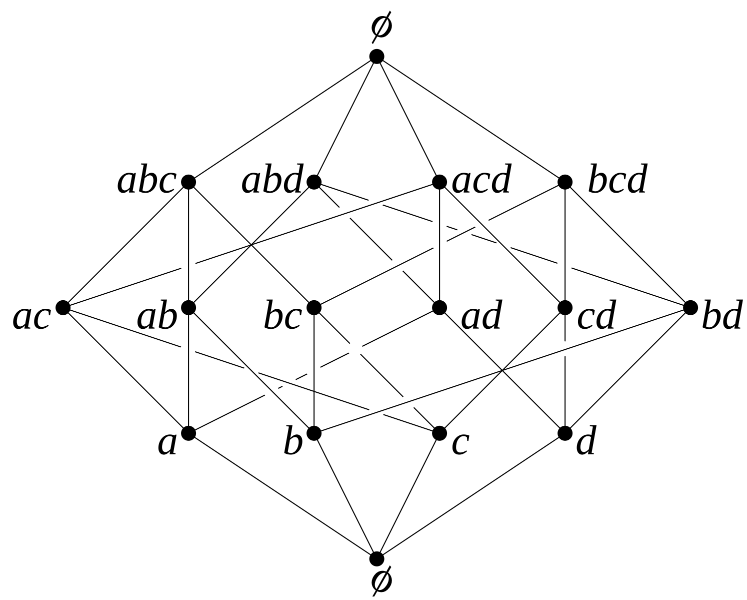
OVERVIEW

- Dualities
- Data structures
  - Quad-Edge
  - Edge-Facet
- Computing Homology
  - Reduction Algorithm
  - Incremental Algorithm

DUALITY



(a) Tetrahedron

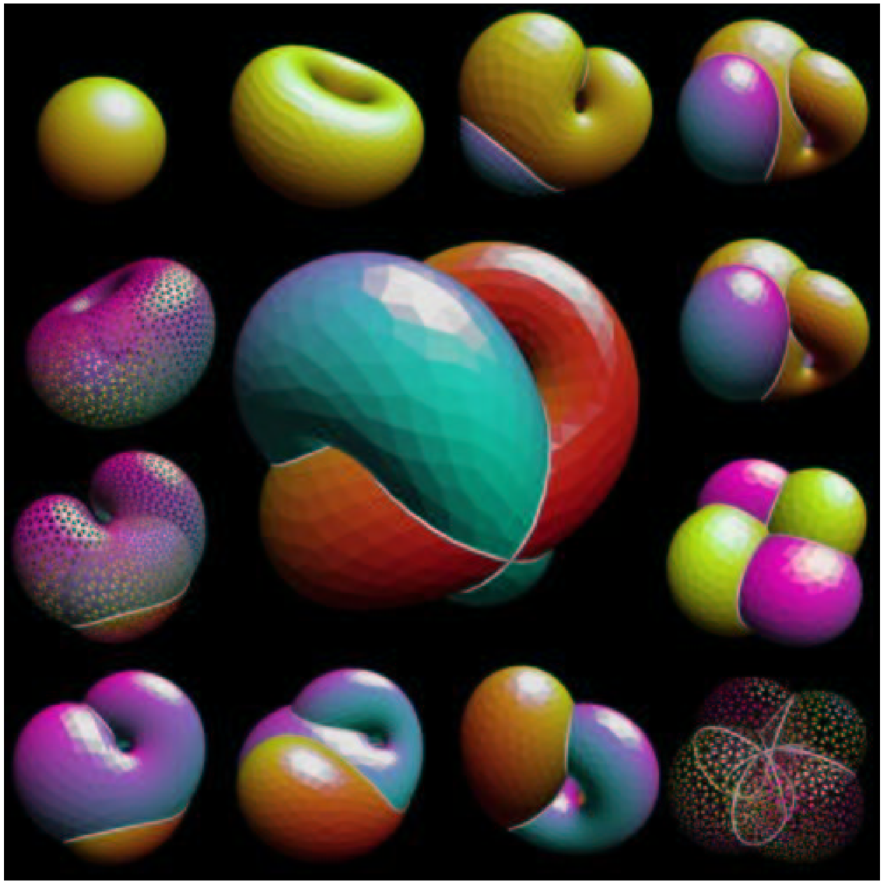


(b) Poset

PLATONIC SOLIDS

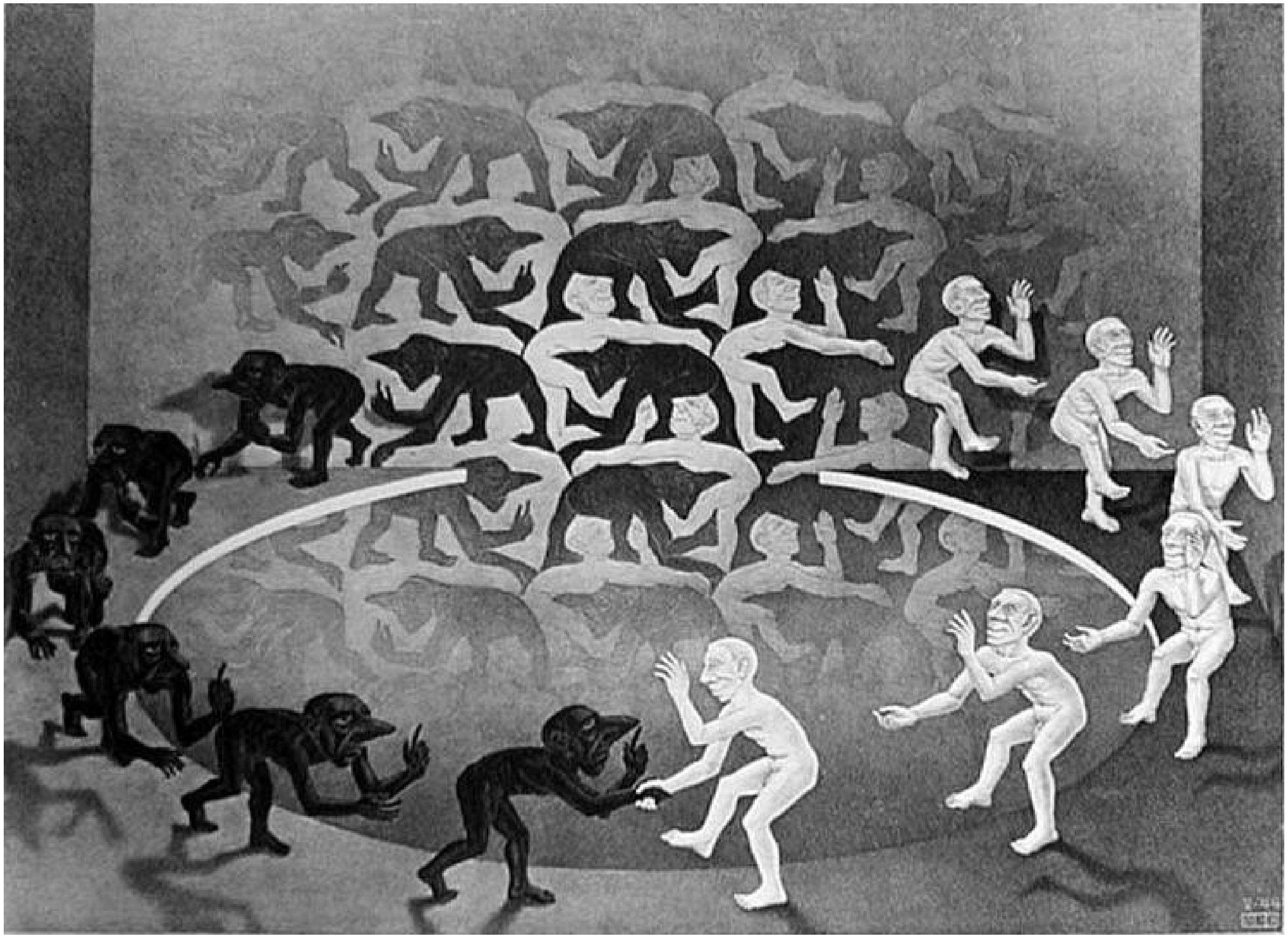
solid	vertices	edges	faces
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
dodecahedron	20	30	12
icosahedron	12	30	20

ORIENTATION



The Optiverse [Sullivan '98]

BACKGROUND-FOREGROUND



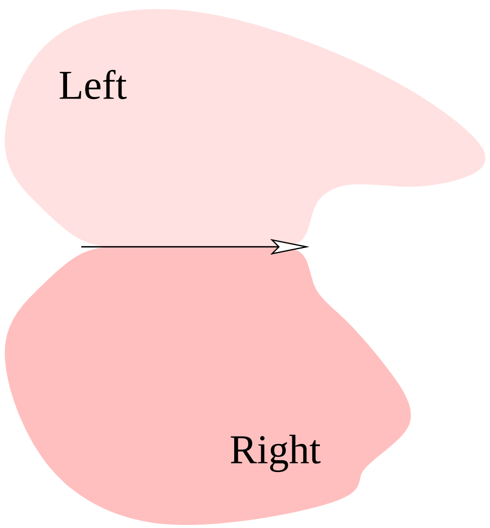
COMPLEMENTARITY

TIME REVERSAL

DUALITY

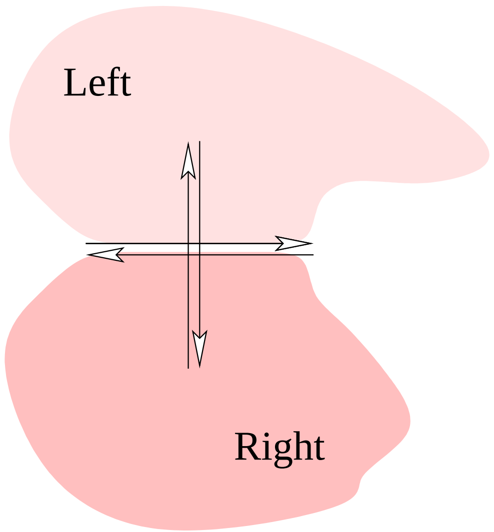
- Orientation: inside vs. outside
- Structure: primal vs. dual (poset vs. upside-down poset)
- Complementarity: background vs. foreground
- Time reversal: forward vs. backward

DIRECTED EDGE



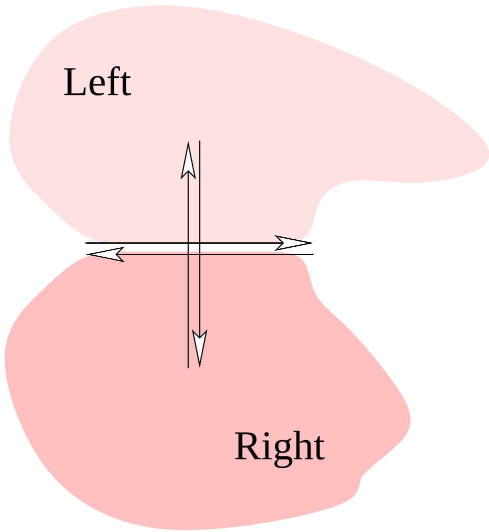
- An edge e has two vertices
- A directed edge goes from Org (e) to Dest (e)
- An edge separates two cells
- Sym (e) goes in the opposite direction

QUAD-EDGE



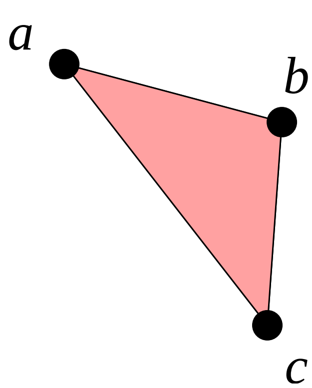
- Rot(e) gives you the dual edge (clockwise) and Tor(e) (counter)
- Edge e stores its number and the next clockwise edge with the same origin: Onext (e)
- A Quad-Edge is Edge[4]: edge, rot, sym, tor
- All operations $O(1)$

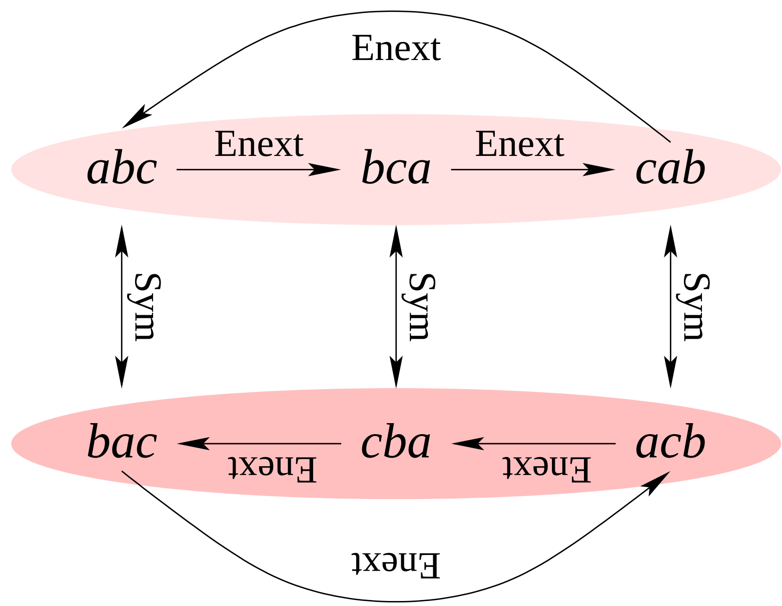
OPERATIONS



- $\text{Oprev}\,(e) = (\text{Rot} \circ \text{Onext} \circ \text{Rot})(e)$
- $\text{Dnext}\,(e) = (\text{Sym} \circ \text{Onext} \circ \text{Sym})(e)$
- $\text{Lnext}\,(e) = (\text{Tor} \circ \text{Onext} \circ \text{Rot})(e)$

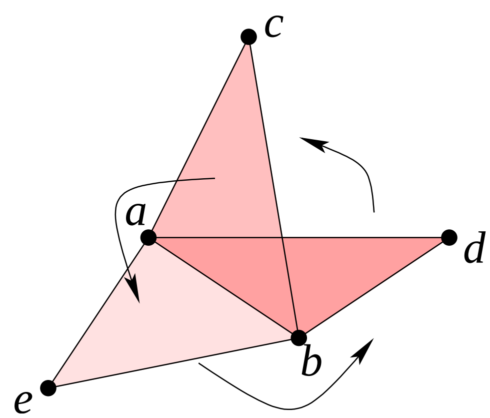
ORIENTED TRIANGLES





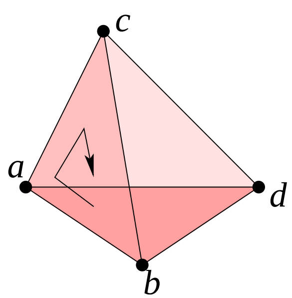
- Sym does one transposition (changes orientation)
- Enext does two transpositions (rotate by 60 degrees clockwise)
- Array of six edge-facets for each triangle
- All operations $O(1)$

FNEXT

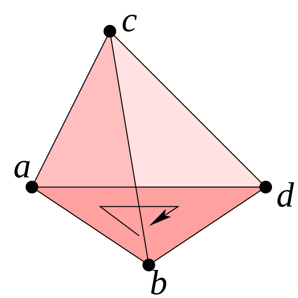


- Fnext $(bac) = bad$
- Fnext $(abd) = abc$
- Fnext $(abc) = abe$
- Each store its Fnext

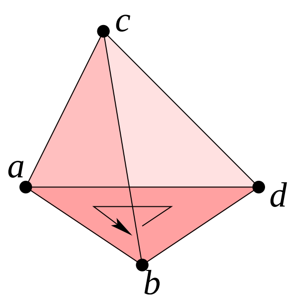
FACES OF A TETRAHEDRON



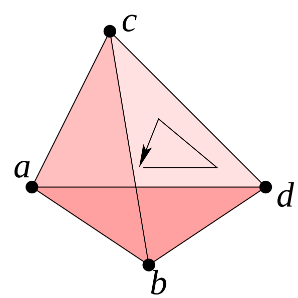
(a) $f = bac$



(b) $\text{Fnext}\,(f) = bad$



(c) $\text{Sym}\,(\text{Fnext}\,(f)) = abd$



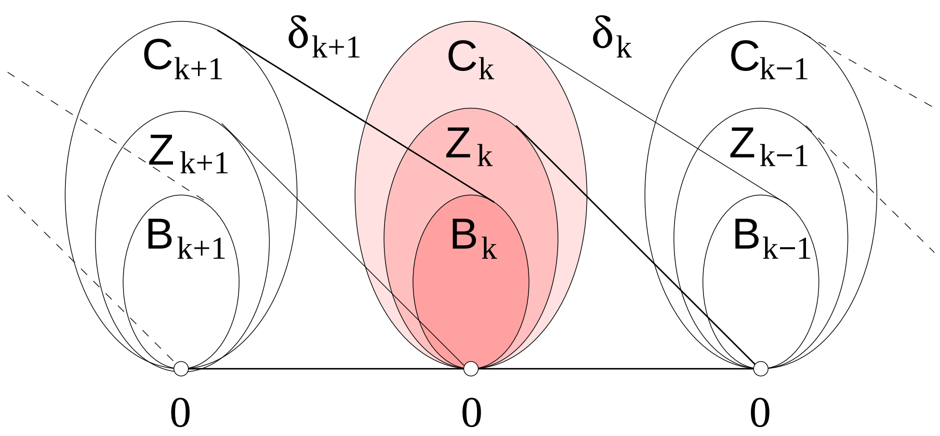
(d) $\text{Sym}\,(\text{Fnext}\,(\text{Enext}\,(f))) = cad$

HOMOLOGY

- The kth homology group is

$$\mathsf{H}_k = \mathsf{Z}_k/\mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

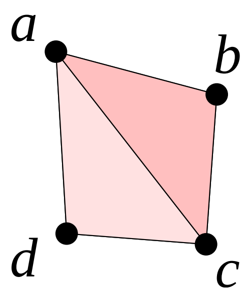
- Compute a basis for $\ker \partial_k$
- Compute a basis for $\operatorname{im} \partial k + 1$



Matrix Representation

- Boundary homomorphism is linear, so it has a matrix
- $\partial_k : \mathsf{C}_k \longrightarrow \mathsf{C}_{k-1}$
- Use oriented simplices as bases for domain and codomain!
- M_k is the standard matrix representation for ∂_k

EXAMPLE



$$M_1 = \left[\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline a & -1 & 0 & 0 & -1 & -1 \\ b & 1 & -1 & 0 & 0 & 0 \\ c & 0 & 1 & -1 & 0 & 1 \\ d & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Elementary Operations

- The elementary row operations on M_k are
  1. exchange row i and row j,
  2. multiply row i by -1,
  3. replace row i by (row i) $+ q$(row j), where q is an integer and $j \neq i$.
- Similar elementary column operations on columns
- Effect: change of bases

DESCRIPTION

- Homology groups are finitely generated abelian.
- (Theorem) Every finitely generated abelian group is isomorphic to product of cyclic groups of the form

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z},$$

- $\beta_k = \beta(\mathsf{H}_k)$
- Torsion coefficients

INTUITION

- How do we find cycles?
- How do we find boundaries?
- What does a free generator correspond to?
- How does a torsional element appear?

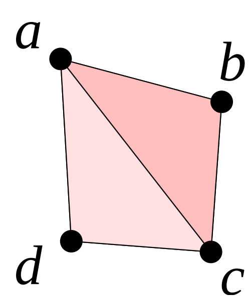
Reduction Algorithm

- Like Gaussian elimination, we keep changing the basis to get to the (Smith) normal form:

$$\tilde{M}_k = \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_{l_k} & \\ \hline & 0 & & 0 \end{array}\right]$$

- $l_k = \text{rank}\, M_k = \text{rank}\, \tilde{M}_k$, $b^i \geq 1$
- $b_i | b_{i+1}$ for all $1 \leq i < l_k$

Reduced Example



$$\tilde{M}_1 = \left[\begin{array}{c|ccccc} & cd & bc & ab & z_1 & z_2 \\ \hline d-c & 1 & 0 & 0 & 0 & 0 \\ c-b & 0 & 1 & 0 & 0 & 0 \\ b-a & 0 & 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- $z_1 = ad - bc - cd - ab$ and $z_2 = ac - bc - ab$ form a basis for Z_1
- $\{d - c, c - b, b - a\}$ is a basis for B_0

Reduced Example

$$M_2 = \left[\begin{array}{c|cc} & abc & acd \\ \hline ac & -1 & 1 \\ ad & 0 & -1 \\ cd & 0 & 1 \\ bc & 1 & 0 \\ ab & 1 & 0 \end{array}\right]$$

$$\tilde{M}_2 = \left[\begin{array}{c|cc} & -abc & -acd + abc \\ \hline ac - bc - ab & 1 & 0 \\ ad - cd - bc - ab & 0 & 1 \\ cd & 0 & 0 \\ bc & 0 & 0 \\ ab & 0 & 0 \end{array}\right]$$

NORMAL FORM

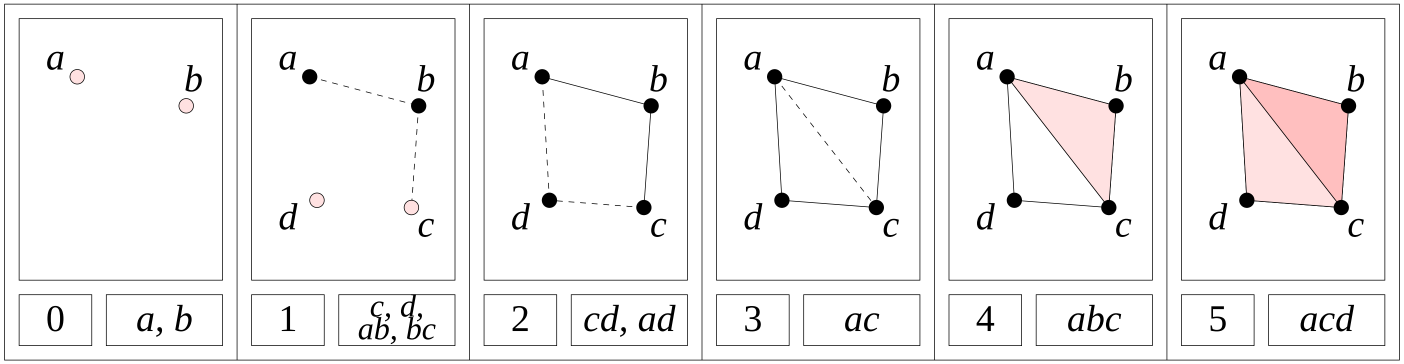
$$\tilde{M}_k = \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_{l_k} & \\ \hline & 0 & & 0 \end{array}\right]$$

- Description:
  1. the torsion coefficients of H_{k-1} are $b_i \geq 1$
  2. $\{e_i \mid l_k + 1 \leq i \leq m_k\}$ is a basis for Z_k. Therefore, $\operatorname{rank} \mathsf{Z}_k = m_k - l_k$.
  3. $\{b_i \hat{e}_i \mid 1 \leq i \leq l_k\}$ is a basis for B_{k-1}. Equivalently, $\operatorname{rank} \mathsf{B}_k = \operatorname{rank} M_{k+1} = l_{k+1}$.
- $\beta_k = \operatorname{rank} \mathsf{Z}_k - \operatorname{rank} \mathsf{B}_k = m_k - l_k - l_{k+1}$

IN $\mathbb{S}^3$

- Algorithm takes $O(m^3)$ operations, but integers can get large
- Subcomplexes are torsion-free, so we don't need the force!
- k-chain: $c = \sum_i n_i[\sigma_i], n_i \in \mathbb{Z}, \sigma_i \in K$
- Different view, n_i are coefficients
- We can multiply, but not divide (in $\mathbb{Z}$)
- We can also change to other coefficients, such as $\mathbb{R}$, $\mathbb{Q}$, etc.
- $\mathbb{Z}_2$ Homology
  - restrict to 0,1, so unoriented simplices
  - $-\sigma = \sigma$
  - Addition is symmetric sum: $c + d = (c \cup d) - (c \cap d)$.

Filtration



- A filtration of a complex K is $\emptyset = K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m = K$.
- A filtration is a partial ordering
- Sort according to dimension
- Break other ties arbitrarily
- Algorithm for $K = \mathbb{S}^3$

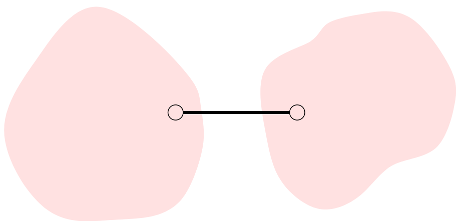
ALEXANDER DUALITY

- Alexander Duality:
  - β_0 measures the number of components of the complex.
  - β_1 is the rank of a basis for the tunnels.
  - β_2 counts the number of voids in the complex.
- An incremental approach:
  - add each simplex in turn
  - check to see if we form a new cycle class or destroy one.

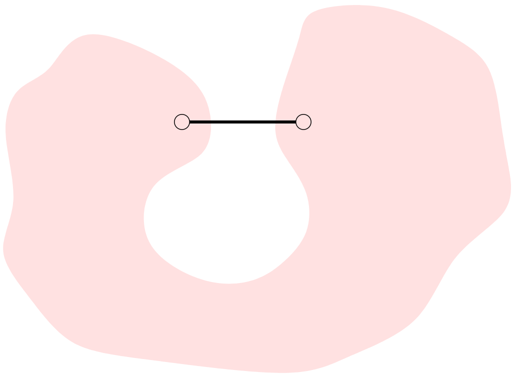
VERTICES

- Vertices always add a new component, so β_0++. Why?
- Union-find data-structure:
  - MAKESET: initializes a set with an item
  - FIND: finds the set an element belongs to
  - UNION: forms the union of two sets
- Very simple to implement
- $O(n)$ space
- Amortized $\alpha(m)$ FIND, UNION
- MAKESET for each vertex

EDGES



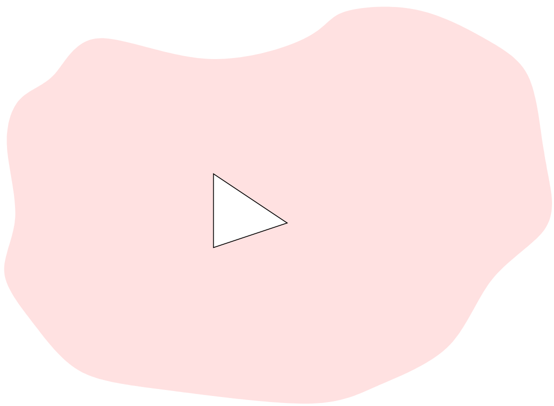
(a) $\beta_0{}^{--}$



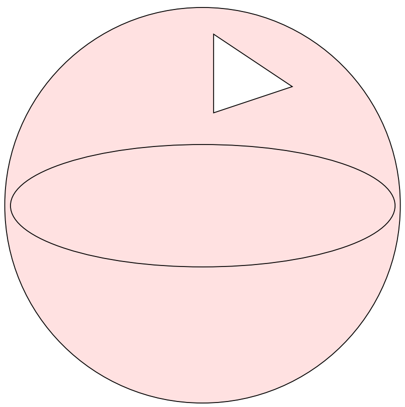
(b) $\beta_1{}^{++}$

- (a) Two FINDs, one UNION
- (b) Two FINDs

TRIANGLES AND TETRAHEDRA



(a) $\beta_1{}^{--}$



(b) $\beta_2{}^{++}$

- Tetrahedra always fill voids, so $\beta_2{}^{--}$

Computing Voids

$$
\begin{array}{ccccccc}
\text{1 void} & \xrightarrow{\text{tri}} & \text{2 voids} & \xrightarrow{\text{tet}} & \text{1 void} & \xrightarrow{\text{tet}} & \text{1 component} \\
\downarrow{\scriptstyle\text{complement}} & & \downarrow & & \downarrow & & {\scriptstyle\text{complement}}\downarrow \\
\text{1 component} & \xrightarrow{\text{tri}} & \text{2 components} & \xrightarrow{\text{tet}} & \text{1 component} & \xrightarrow{\text{tet}} & \text{1 void} \\
\downarrow{\scriptstyle\text{dual}} & & \downarrow & & \downarrow & & {\scriptstyle\text{dual}}\downarrow \\
\text{1 vertex} & \xleftarrow{\text{edge}} & \text{2 vertices} & \xleftarrow{\text{vertex}} & \text{1 vertex} & \xleftarrow{\text{vertex}} & \emptyset
\end{array}
$$

- We can always look at the complement in $\mathbb{S}^3$
- Dualize to get vertices and edges
- Reverse time to get to Union-Find

INCREMENTAL ALGORITHM

- Three passes:
  - One pass to identify negative edges
  - One reverse dual pass on the complement space to get positive triangles
  - One pass to compute them all (the Betti numbers)
- $O(m\alpha(m))$

dim	0	1	2
0	++		
1	−−	++	
2		−−	++
3			−−