An Introduction to the Theory of Knots

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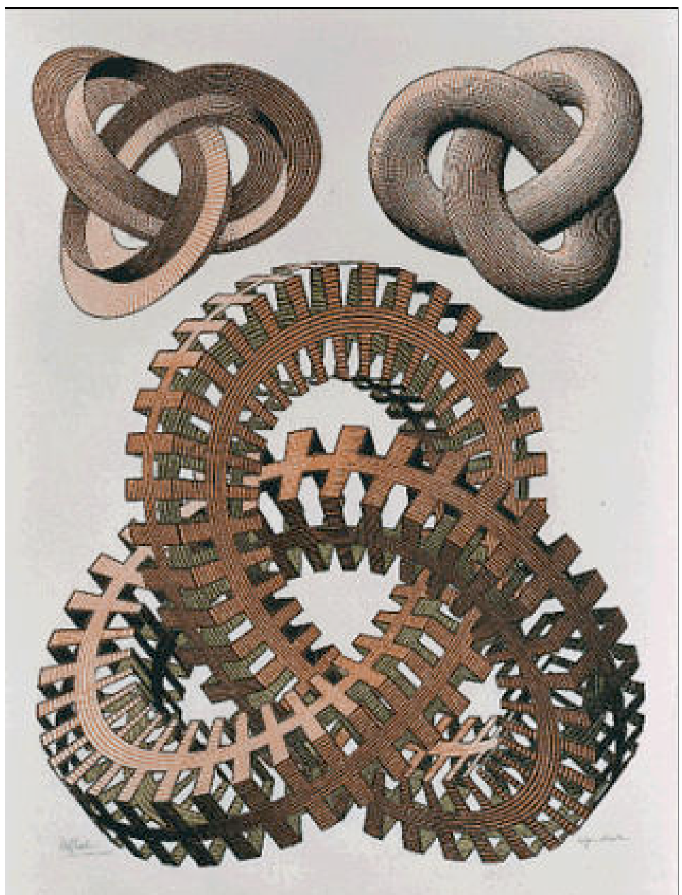

Figure 1: Escher's Knots, 1965

1 Knot Theory

Knot theory is an appealing subject because the objects studied are familiar in everyday physical space. Although the subject matter of knot theory is familiar to everyone and its problems are easily stated, arising not only in many branches of mathematics but also in such diverse fields as biology, chemistry, and physics, it is often unclear how to apply mathematical techniques even to the most basic problems. We proceed to present these mathematical techniques.

1.1 Knots

The intuitive notion of a knot is that of a knotted loop of rope. This notion leads naturally to the definition of a knot as a continuous simple closed curve in $\mathbb{R}^3$. Such a curve consists of a continuous function $f : [0, 1] \to \mathbb{R}^3$ with $f(0) = f(1)$ and with $f(x) = f(y)$ implying one of three possibilities:

1. $x = y$
2. $x = 0$ and $y = 1$
3. $x = 1$ and $y = 0$

Unfortunately this definition admits pathological or so called wild knots into our studies. The remedies are either to introduce the concept of differentiability or to use polygonal curves instead of differentiable ones in the definition. The simplest definitions in knot theory are based on the latter approach.

Definition 1.1 (knot) *A knot is a simple closed polygonal curve in* $\mathbb{R}^3$.

The ordered set $(p_1, p_2, \ldots, p_n)$ defines a knot; the knot being the union of the line segments $[p_1, p_2], [p_2, p_3], \ldots, [p_{n-1}, p_n]$, and $[p_n, p_1]$.

Definition 1.2 (vertices) *If the ordered set* $(p_1, p_2, \ldots, p_n)$ *defines a knot and no proper ordered subset defines the same knot, the elements of the set,* p_i*, are called the vertices of the knot.*

Projections of a knot to the plane allow the representation of a knot as a knot diagram. Certain knot projections are better than others as in some projections too much information is lost.

Definition 1.3 (regular projection) *A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.*

Theorem 1.1 *If a knot does not have a regular projection then there is an equivalent knot that does have a regular projection.*

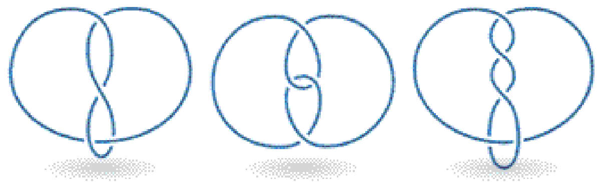

Figure 2: Three knot diagrams for the figure - eight knot.

A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot undercrosses the other part.

Informally, an orientation of a knot can be thought of as a direction of travel around the knot.

Definition 1.4 (oriented knot) *An oriented knot consists of a knot and an ordering of its vertices. The ordering must be chosen so that it determines the original knot. Two orderings are considered equivalent if they differ by a cyclic permutation.*

The orientation of a knot on a knot diagram is represented by placing coherently directed arrows.

The connected sum of two knots, K_1 and K_2, is formed by removing a small arc from each knot and then connecting the four endpoints by two new arcs in such a way that no new crossings are introduced, the result being a single knot, $K = K_1 \# K_2$.

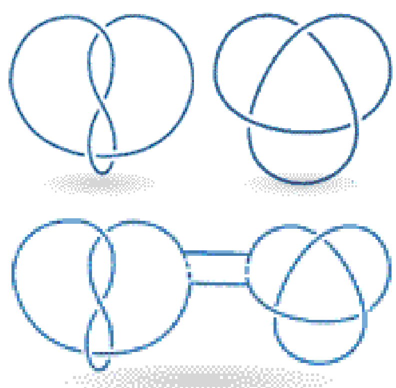

Figure 3: Connected sum of the figure - eight knot and the trefoil knot.

The notion of equivalence of knots is based on their knot diagrams and the following theorem.

Theorem 1.2 *If knots K and J have identical diagrams, then they are equivalent.*

1.2 Equivalence

The notion of equivalence satisfies the definition of an equivalence relation; it is reflexive, symmetric, and transitive. Knot theory consists of the study of equivalence classes of knots. In general it is a difficult problem to decide whether or not two knots are equivalent or lie in the same equivalence class, and much of knot theory is devoted to the development of techniques to aid in this decision.

A Reidemeister move is an operation that can be performed on the diagram of a knot whithout altering the corresponding knot.

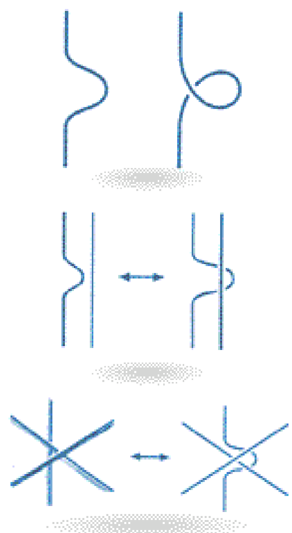

Figure 4: Type I, type II, and type III Reidemeister moves.

They correspond to the simplest changes in a diagram that occur when a knot is deformed. Although each of these moves changes the diagram, they do not change the knot represented by the diagram.

Theorem 1.3 *If two knots are equivalent, their diagrams are related by a sequence of Reidemeister moves.*

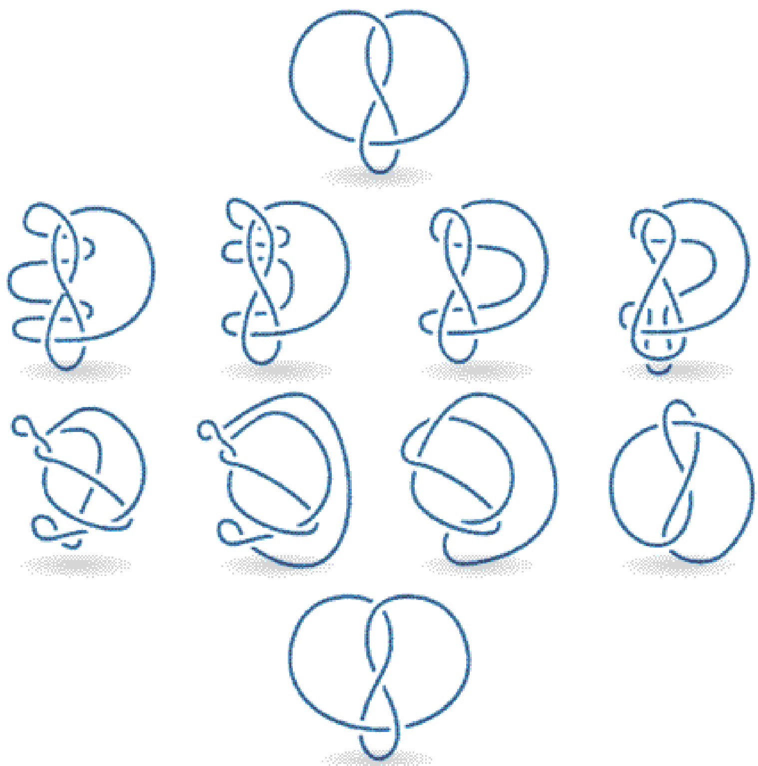

Figure 5: The figure - eight knot is equivalent to its mirror image. A knot with this property is called *amphicheiral*.

Definition 1.5 (colorable) *A knot diagram is called colorable if each arc can be drawn using one of three colors in such a way that:*

1. *At least two of the colors are used.*
2. *At each crossing either three different colors come together or all the same color comes together.*

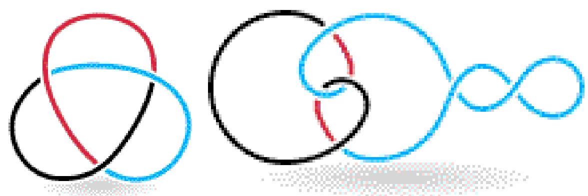

Figure 6: The trefoil knot is colorable.

Theorem 1.4 *If a diagram of a knot K, is colorable, then every diagram of K is colorable.*

Definition 1.6 (colorable) *A knot is called colorable if its diagrams are colorable.*

Clearly the unknot is not colorable because its standard projection cannot be colored. On the other hand, every projection of the trefoil knot is colorable. These two observations in conjunction with the theorem above leads to the most basic fact of knot theory – the existence of nontrivial knots, knots other than the unknot. Furthermore any colorable knot is nontrivial.

The concept of colorability can be generalized by introducing the concept of a mod$p$ labeling.

Definition 1.7 (mod$p$ labeling) *A knot diagram can be labeled* mod$p$ *if each edge can be labeled with an integer from* 0 *to* $p-1$ *such that*

1. *At least two labels are distinct.*
2. *At each crossing the relation* $2x - y - z = 0 \pmod p$ *holds, where x is the label on the overcrossing and y and z the other two labels.*

Theorem 1.5 *If some diagram for a knot can be labeled* mod$p$ *then every diagram for that knot can be labeled* mod$p$.

The concept of mod$p$ labelings is an interesting generalization of colorability since the question of whether a diagram can be labeled mod$p$ can be reduced to a problem in linear algebra; whether there is a mod$p$ solution to a system of algebraic equations.

The concept of mod$p$ labelings can be generalized by introducing the concept of labeling knots with elements of a group.

Definition 1.8 (group labeling) *A labeling of an oriented knot diagram with elements of a group consists of assigning an element of the group to each arc of the diagram, subject to the following two conditions.*

1. *At each crossing of the diagram three arcs appear, each of which should be labeled with an element from the group. The label of the arc that passes under the crossing must be conjugate to the one that emerges from the crossing via the label on the overcrossing.*
2. *The labels must generate the group.*

Theorem 1.6 *If a diagram for a knot can be labeled with elements from a group G, then any diagram of the knot can be so labeled with elements from that group, regardless of the choice of orientation.*

Finding a labeling of a knot using elements of a group is quite difficult. Fortunately we can proceed systematically by reducing the problem of finding a labeling for the knot to that of solving equations in a group; once a few labels are chosen the rest are determined. Furthermore the procedure above results in a collection of variables and relations of the form $r_i = 1$ from which we can form a group presentation. The resulting group is called the group of the knot. Although the group depends on the arbitrary choices made, it can be proved that all groups that arise this way for a given knot are isomorphic. Another significant result in the subject, Dehn's Lemma, states that if a knot group is isomorphic to $\mathbb{Z}$, then the knot is trivial.

We shift the focus of our study of knot theory from the methods based on knot diagrams to those based on surfaces. This shift allows for the use of geometric and topological techniques. It is motivated by a theorem stating that for any knot there is some surface having that knot as its boundary. Furthermore, there is an algorithm for its explicit construction, and the resulting surface is called a Seifert surface for the knot.

Theorem 1.7 *Every knot is the boundary of an orientable surface.*

The theorems below gives a homeomorphism classification for the surfaces.

Theorem 1.8 (Classification I) *Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.*

Theorem 1.9 (Classification II) *Two disks with bands attached are homeomorphic if and only if the following three conditions are met:*

1. *They have the same number of bands.*
2. *They have the same number of boundary components.*
3. *Both are orientable or both are nonorientable.*

In the same spirit as studying the surface of a knot, we may study the complement of the knot in three space, $\mathbb{R}^3 - K$, and form its fundamental group. The use of the fundamental group allows the definition of algebraic quantities without reference to diagrams for the knot. This framework also brings into play the powerful techniques of algebraic topology, for instance, homology theory.

Below we define the classical and most natural invariants in the study of knots. They are ways of associating integers to knots.

Definition 1.9 (crossing number) *The crossing number of a knot K, denoted c(K), is the least number of crossings that occur, ranging over all possible diagrams.*

Definition 1.10 (unknotting number) *The unknotting number of a knot K, denoted u(K), is the least number of crossing changes that are required for the knot to become unknotted, ranging over all possible diagrams.*

Definition 1.11 (bridge number) *The bridge number of a knot K, denoted b(K), is the least number of bridges that occur, ranging over all possible diagrams. A bridge is considered to be an arc between two undercrossings with no undercrossings inbetween and at least one overcrossing.*

Although beyond the scope of this leisurely introduction to knot theory, one of the most successfull and interesting ways to tell knots apart is through the various knot polynomials, of which there is an incredible variety.

The definition of a prime knot and the prime decomposition theorem can now be presented.

Definition 1.12 (prime knot) *A knot is called prime if for any decomposition as a connected sum, one of the factors is unknotted.*

Theorem 1.10 (Prime Decomposition Theorem) *Every knot can be decomposed as the connected sum of nontrivial prime knots. If $K = K_1 \# K_2 \# \cdots \# K_n$, and $K = J_1 \# J_2 \# \cdots \# J_m$, with each K_i and J_i nontrivial prime knots, then m = n, and, after reordering each K_i is equivalent to J_i*

A Appendix

Number of Crossings	Number of Prime Knots
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165
11	552
12	2,176
13	9,988
14	46,972
15	253,293
16	1,388,705

Table 1: Table of knots through sixteen crossings. Mirror images are excluded from the count.

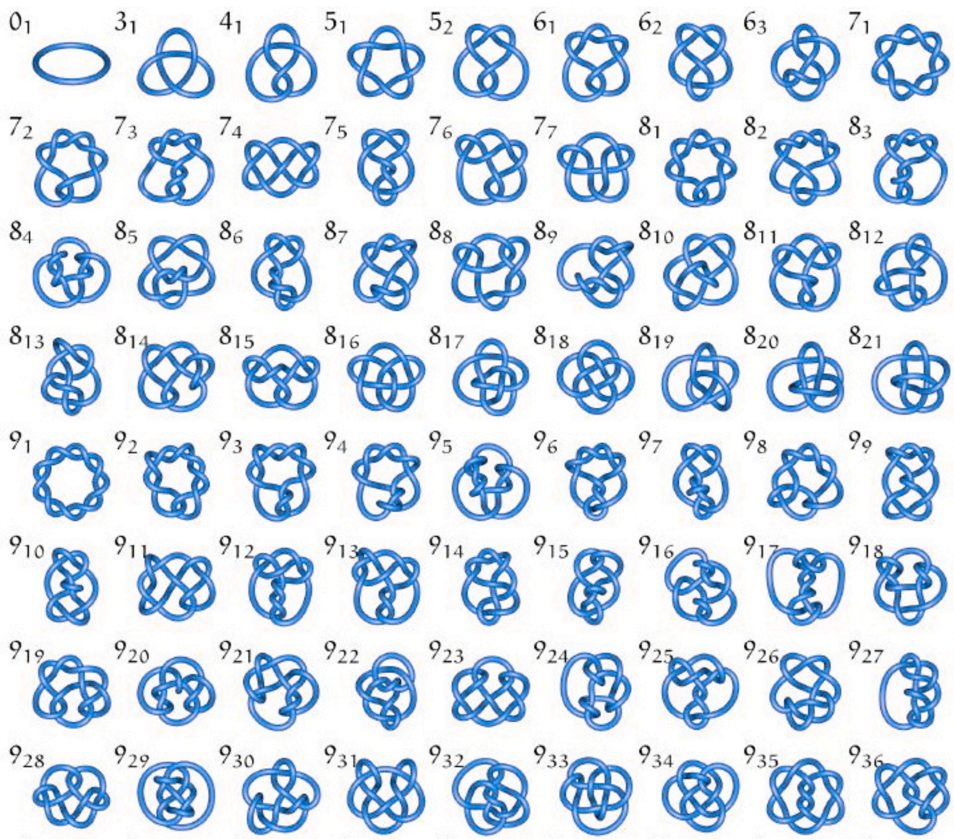

Figure 7: Table of knots through eight crossings, and most nine crossing knots.

Acknowledgements

Most of the material is from Livingston [3] and Adams [1].

References

[1] Adams, Colin C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* W. H. Freeman and Company, New York, New York, 2001.

[2] Crowell, R. H. and Fox, R. H. *Introduction to Knot Theory.* Graduate Texts in Mathematics, Volume Fifty Seven. Springer - Verlag, New York - Heidelberg - Berlin, 1977.

[3] Livingston, C. *Knot Theory.* The Carus Mathematical Monographs, Volume Twenty Four. The Mathematical Association of America, Washington, D. C., 1993.