Euclidean Distance Maps and Eikonal Equations

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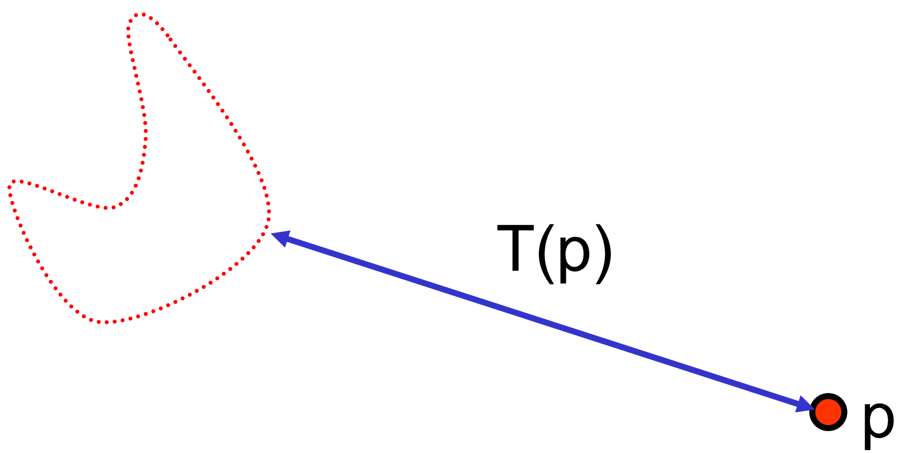
Papers

- H.K. Zhao, "*A fast sweeping method for eikonal equations*".
- P.E. Danielsson, "*Euclidean distance mapping*".
- H. Pottmann, S. Leopoldseder, H.K. Zhao, "*The d^2-tree: A hierarchical representation of the squared distance function*".
- Book: R. Kimmel, "*Numerical Geometry of Images*", Springer-Verlag.

Distance Map

- Let S be a set of source points (representing a curve, surface, object), and D the domain of interest
- A distance map is a function $T: D \rightarrow R_+$, s.t.

$$T(p) = \inf_{q \in S} \|p - q\|_{L^2}$$



Computing Distance Maps

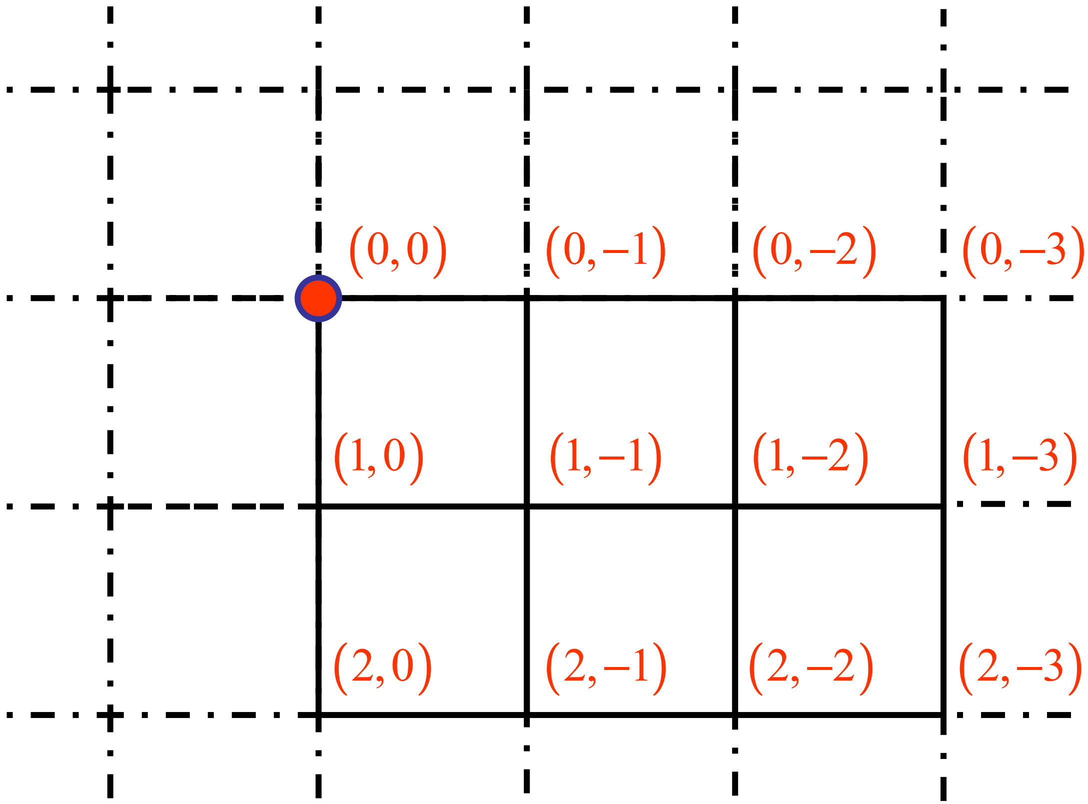
- Q1: So how do we compute distance maps?
- A1: For each point of interest in the domain *D*, scan all source points in *S* and find the closest one.
- Drawback: Will take forever…
- Q2: So how do we compute distance maps and get a result in our lifetime?
- A2: Sweeps with alternating directions.

Danielsson's Algorithm

- 2-D case: For each point we store the *(x,y)* offset to the closest point.
- Initially, all offsets of points in *S* are (0,0) and offsets of points not in *S* are (∞,∞).
- Scan the image 4 times in alternating directions (up/down, left/right).
- Each point checks the values of its four closest neighbors, and updates its own value accordingly.

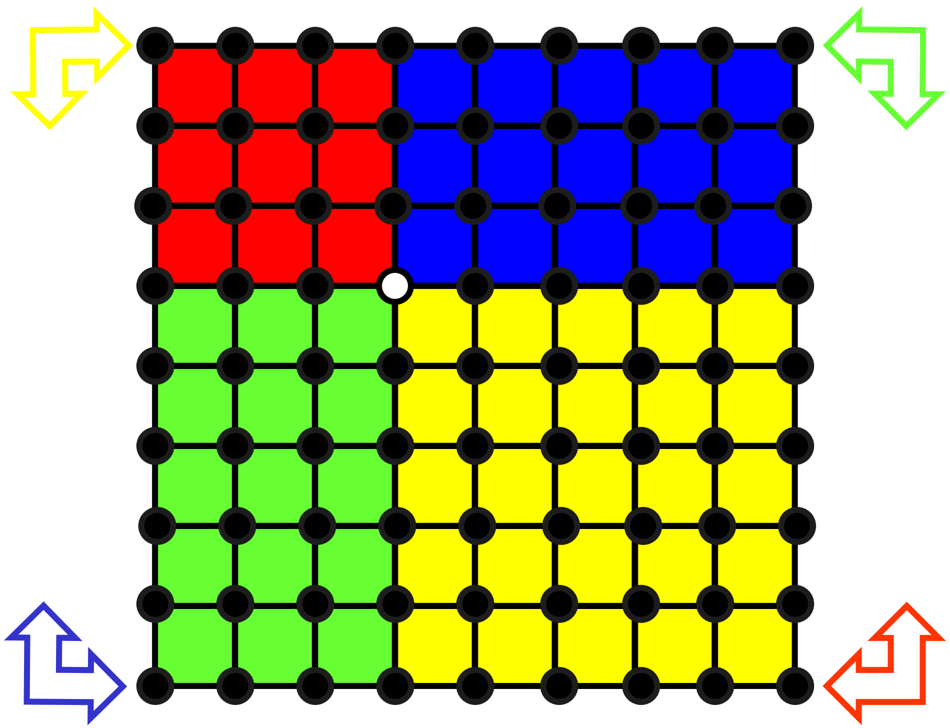
Danielsson's Algorithm

- Example: One source point



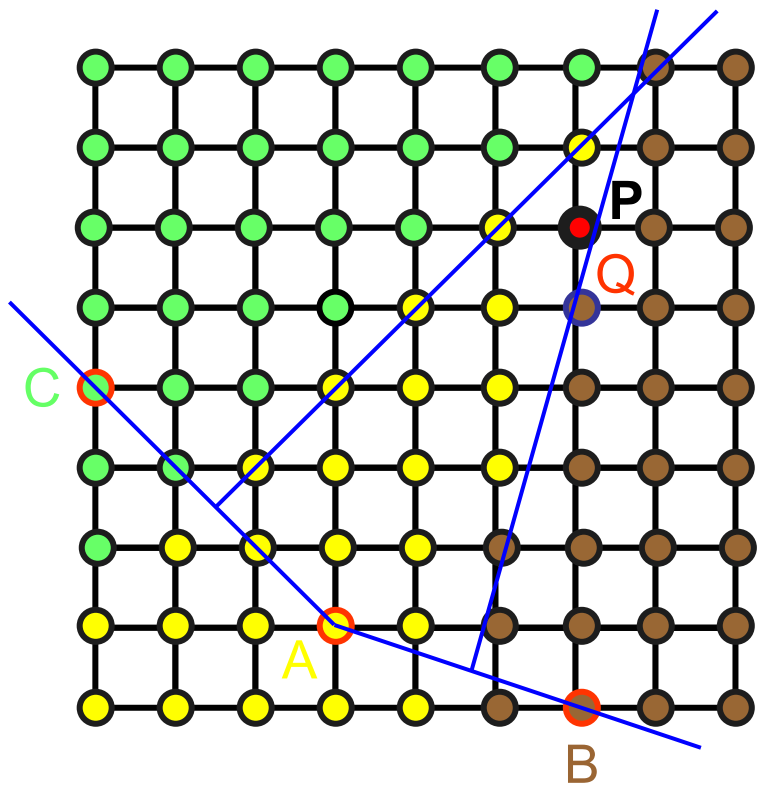
Danielsson's Algorithm

- How does it work?



Danielsson's Algorithm

- What can go awry?



$d(Q,A) = 5$

$d(Q,B) = 5$

Danielsson finds:

$T(P) = 6$

But actually:

$T(P) = \sqrt{35} = 5.916$

Danielsson's Algorithm

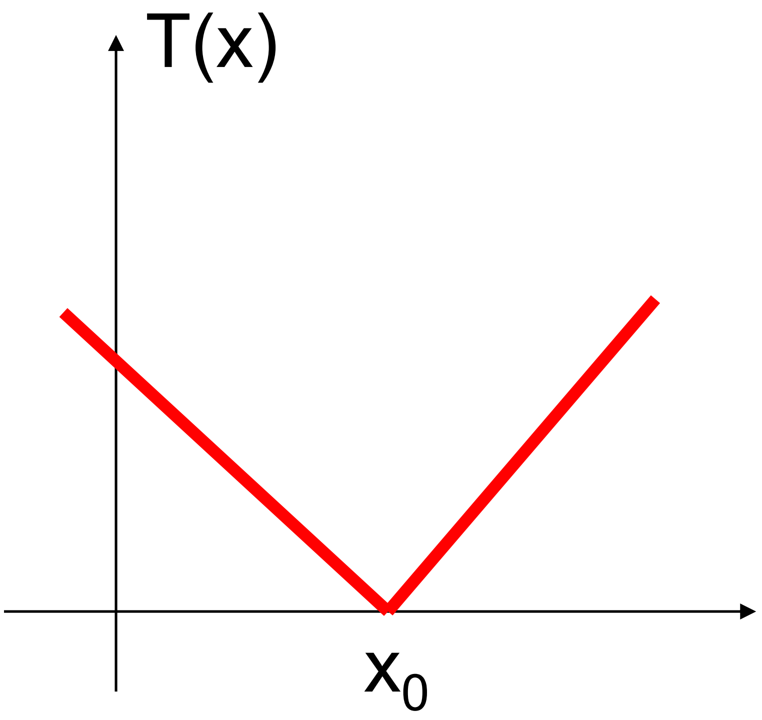
- This argument can be made more precise, to show that the error in the approximation is bounded by *0.29h*, where *h* is the mesh size.
- Improvement: use 8 connectivity instead of 4 connectivity.
- The error bound then becomes *0.076h*.
- However, it involves twice as much work.

Extension to higher dimensions

- This method can be easily extended to other dimensions:
- 1-D: 2 sweeps (left, right)
- 2-D: 4 sweeps (left/right, up/down)
- 3-D: 8 sweeps
- n-D: 2^n sweeps.

Distance Maps and Eikonal Equations

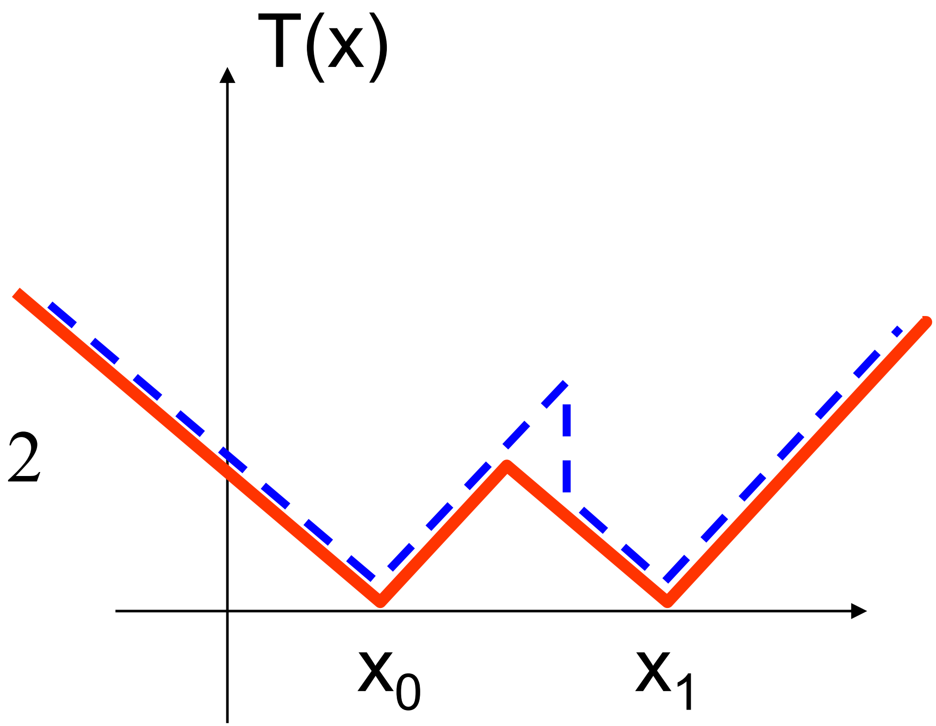
- 1 source point (x_0).
- $T(x) = |x - x_0|$
- $\left|\partial_x T(x)\right| = 1$, except at x_0



Distance Maps and Eikonal Equations

- 2 source points (x_0, x_1).
- $T(x)=\min\{|x-x_0|, |x-x_1|\}$
- $|\partial_x T(x)| = 1,$

  except at $x_0, x_1, (x_0 + x_1)/2$
- The dashed line also satisfies $|\partial_x T(x)| = 1,$ in all but 3 points.



Distance Maps and Eikonal Equations

- So we have that for 1D distance maps:

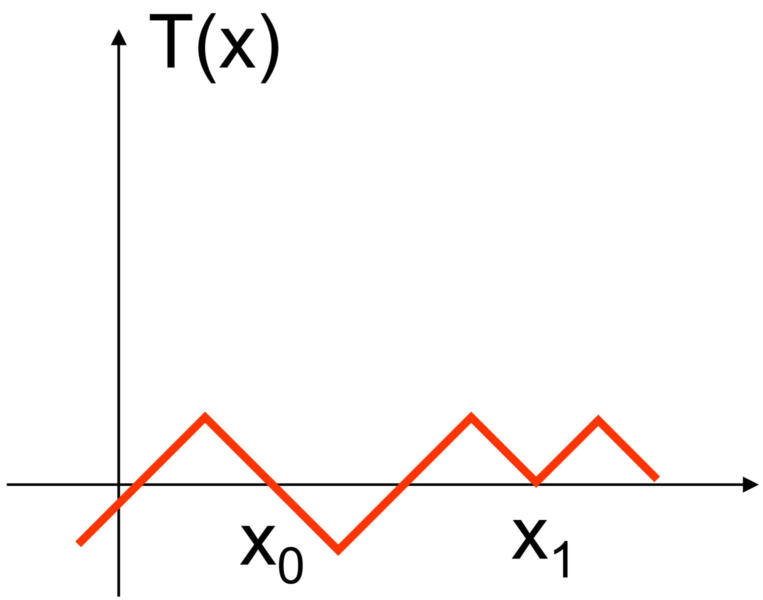
$$\begin{cases} \left|\partial_x T(x)\right| = 1, & x \in \Omega \\ T(x) = 0, & x \in \Gamma \end{cases}$$

- However, the converse is not necessarily true.
- Since the Eikonal equation does not uniquely specify a weak solution, we need to look for a specific solution – a viscosity solution or entropy solution

Discretizing the 1D differential operator

- Backward differencing:

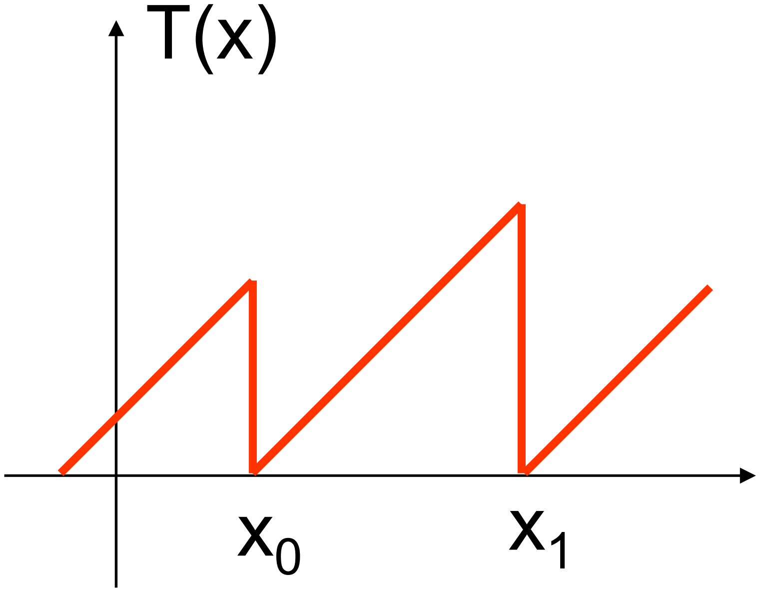
$$\left|\partial_x T(x)\right| = \left|\frac{T_i - T_{i-1}}{h}\right|$$



Discretizing the 1D differential operator

- Truncated Backward differencing:

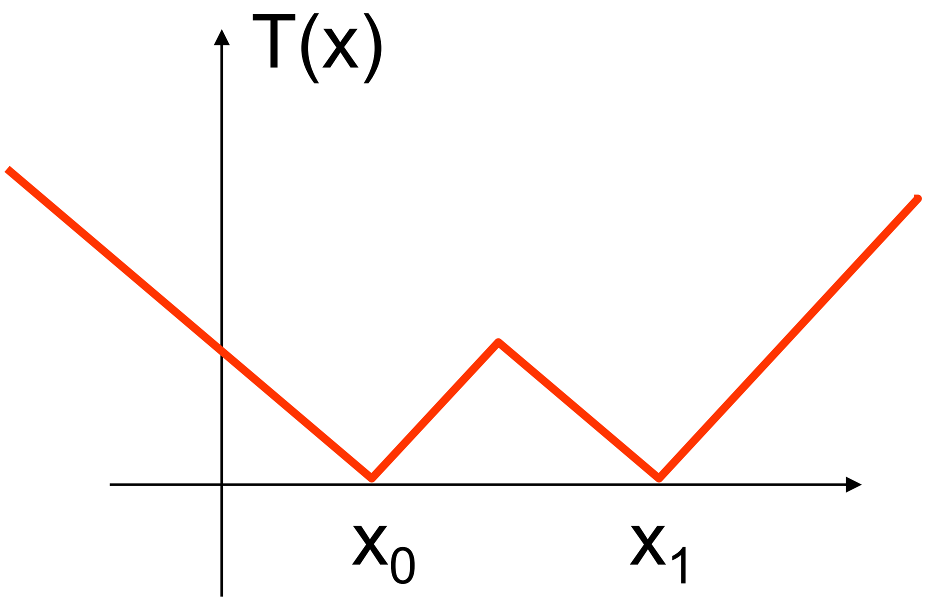
$$\left|\partial_x T(x)\right| = \left[\frac{T_i - T_{i-1}}{h}\right]^+$$



Discretizing the 1D differential operator

- Symmetrized differencing:

$$\left|\partial_x T(x)\right| = \left[\frac{T_i - T_{i-1}}{h}, \frac{T_i - T_{i+1}}{h}\right]^+$$



Discretizing the 1D differential operator

- We can rewrite this scheme as

$$\left|\partial_x T(x)\right| = \left[\frac{T_i - \min\{T_{i-1}, T_{i+1}\}}{h}\right]^+$$

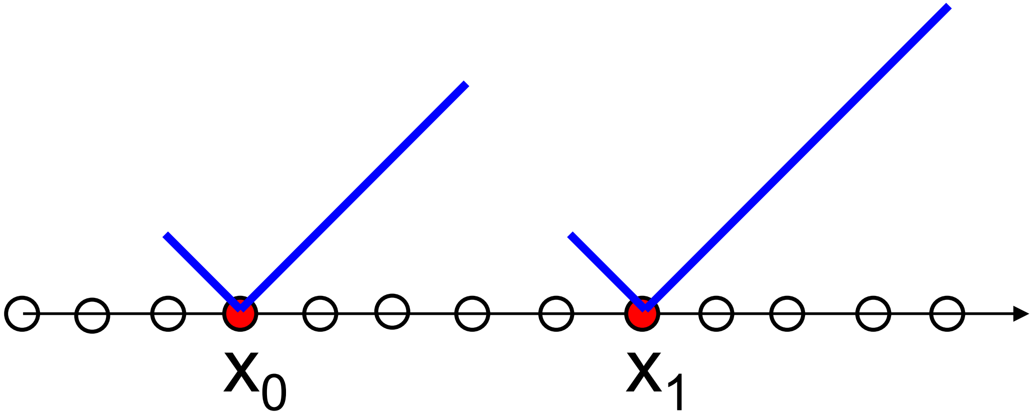
- This numerical approximation is known as an upwind scheme, since it corresponds with the direction of information flow.
- Enforces causality.
- Retrieves the viscosity solution.

Updating order (1D case)

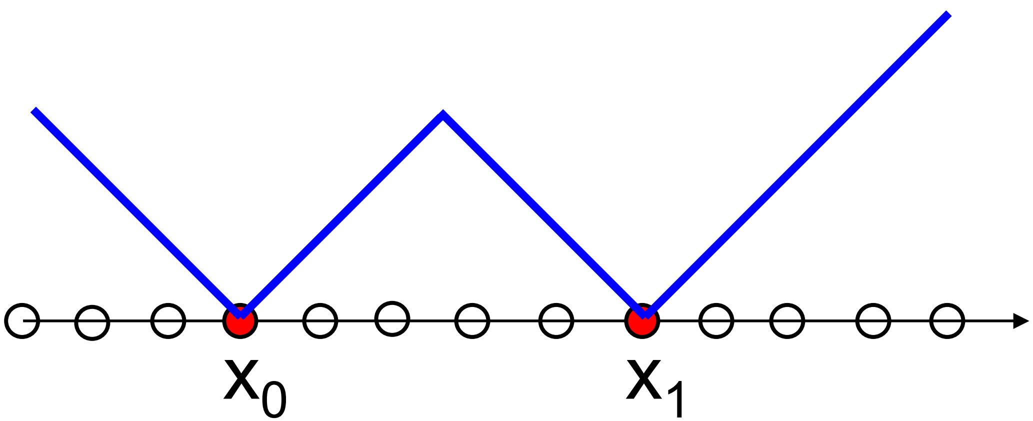
- Q: In which order should we scan the grid?
- A1: We can successively scan it from left to right. In the worst case scenario we will need N scans to converge.
- A2: Do a left-to-right sweep, followed by a right-to-left sweep. Convergence after 2 scans.
- Why? Because the distance value at any grid point can be computed exactly from its left or right neighbor.

Updating order (1D case)

Left-to-right sweep



Right-to-left sweep



n-D Eikonal Equations

- In a more general n-dimensional setting, the Eikonal equation becomes

$$\begin{cases} \left|\nabla T(\vec{x})\right| = F(\vec{x}), \quad \vec{x} \in R^n \\ T(\vec{x}) = 0, \quad \vec{x} \in \Gamma \subset R^n \end{cases}$$

- In 2D, the upwind difference scheme (a.k.a Godunov's scheme) has the form

$$\left(\left[T_{i,j} - \min\left\{T_{i-1,j}, T_{i+1,j}\right\}\right]^+\right)^2 + \left(\left[T_{i,j} - \min\left\{T_{i,j-1}, T_{i,j+1}\right\}\right]^+\right)^2 = h^2 F_{i,j}{}^2$$

Numerical Solution in 2D

- Initialization: $T(x) = 0$ for points in or near the source point set. Other points are assigned large positive values.
- Updating: Gauss-Seidel iterations.
- Apply Danielsson's algorithm to the Gauss-Seidel update scheme, i.e. use 4 sweeps with alternating directions (left/right, up/down).
- As we've seen before, for the case of 1 source point, 4 sweeps with alternating directions recover the exact distance function. In n-D, 2^n sweeps are required.
- When we have more than 1 source point, more than 2^n iterations may be needed for convergence.

A More General Analysis

- We consider the n-D Eikonal equation with $F = 1$, i.e. for recovering distance functions.
- Key Results:
  - For a single source point x_0, the numerical solution $T_h(x)$ converges in 2^n sweeps and satisfies

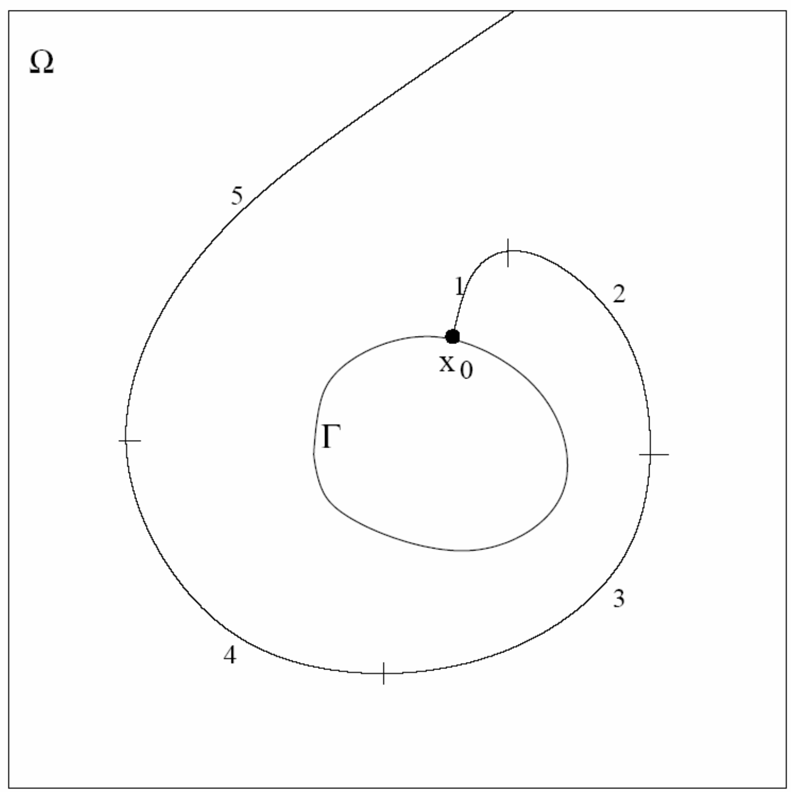
$$d(x) \le T_h(x) \le d(x) + O(h \log h)$$

  - Let $S_h(x)$ denote the solution of the discrete Eikonal equation. For an arbitrary set of source points (not necessarily discrete), the numerical solution $T_h(x)$ after 2^n sweeps satisfies

$$S_h(x) \le T_h(x) \le d(x) + O(h \log h)$$

A More General Analysis

- Q: What happens when *F* is arbitrary?
- A: The number of iterations needed is no longer constant. It depends on the geometric structure of *F*.



Hierarchical Squared Distance Function

- We will now see how to use the fast sweeping algorithm in a hierarchical framework, to estimate the squared distance function of a surface.
- We assume we are given a triangulated surface M.
- The algorithm consists of the following 3 steps:
  1. Construct an octree encompassing the surface M.
  2. Use the fast sweeping algorithm to compute distances of corner points of cubes in the octree to the surface M.
  3. Generate a d^2-tree, which is an octree representation of a piecewise quadratic approximation of the squared distance function of M.

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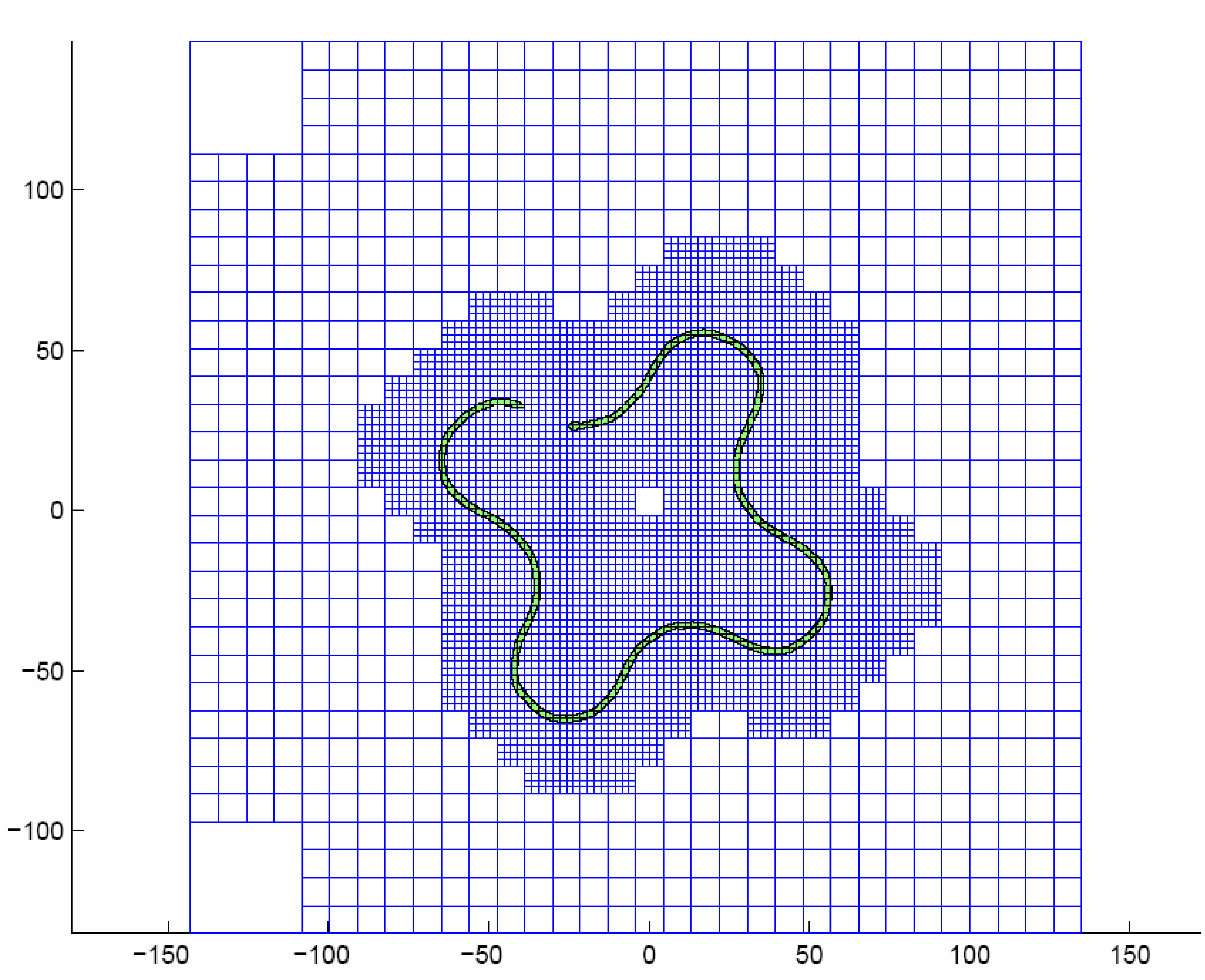
1. Constructing an Octree

- Start with a cube that encloses the object M.
- At level L (starting from L=0), subdivide twice, to get to level L+2.
- Continue in this fashion until we get to level L_{max}, which is a precision parameter of the algorithm.
- Extending the subdivision: For each level L, certain cells C_j^L are already subdivided to level L+2. For each such C_j^L, subdivide all its “neighboring” cells to level L+2.

1.Constructing an Octree

- Example: A planar slice through the octree



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  - ✓ Construct an octree encompassing the surface M.
  - 2. **Use the fast sweeping algorithm to compute distances of corner points of cubes in the octree to the surface M.**

2. Computing a distance function

- We apply the fast sweeping algorithm to compute distance values of corners of cubes, in a multilevel fashion, starting from the finest level and going up.
- Initialization: Run through all triangles of M that intersect a cube of level L_{max}, and compute the distance to the corner points exactly.
- At each level (L_{max}-2..0): Initialize with distances from finer level, and apply the sweeping algorithm on cubes of the current level.

2. Computing a distance function

- Q: How to do raster scans on a tree structure?
- A: Sort the cubes according to the order of the raster scan.
- Suppose all cells of level L are sorted in a list A_L. To sort level L+2, start with the list A_L, remove from it all cells that are not subdivided to level L+2, and sort the children of the remaining cells.

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  - ✓ Construct an octree encompassing the surface M.
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  3. **Generate a d^2-tree, which is an octree representation of a piecewise quadratic approximation of the squared distance function of M.**

3. Computing the d^2-tree

- We shall now construct a new octree, d^2-tree, that will store for each cube a quadratic function of the form

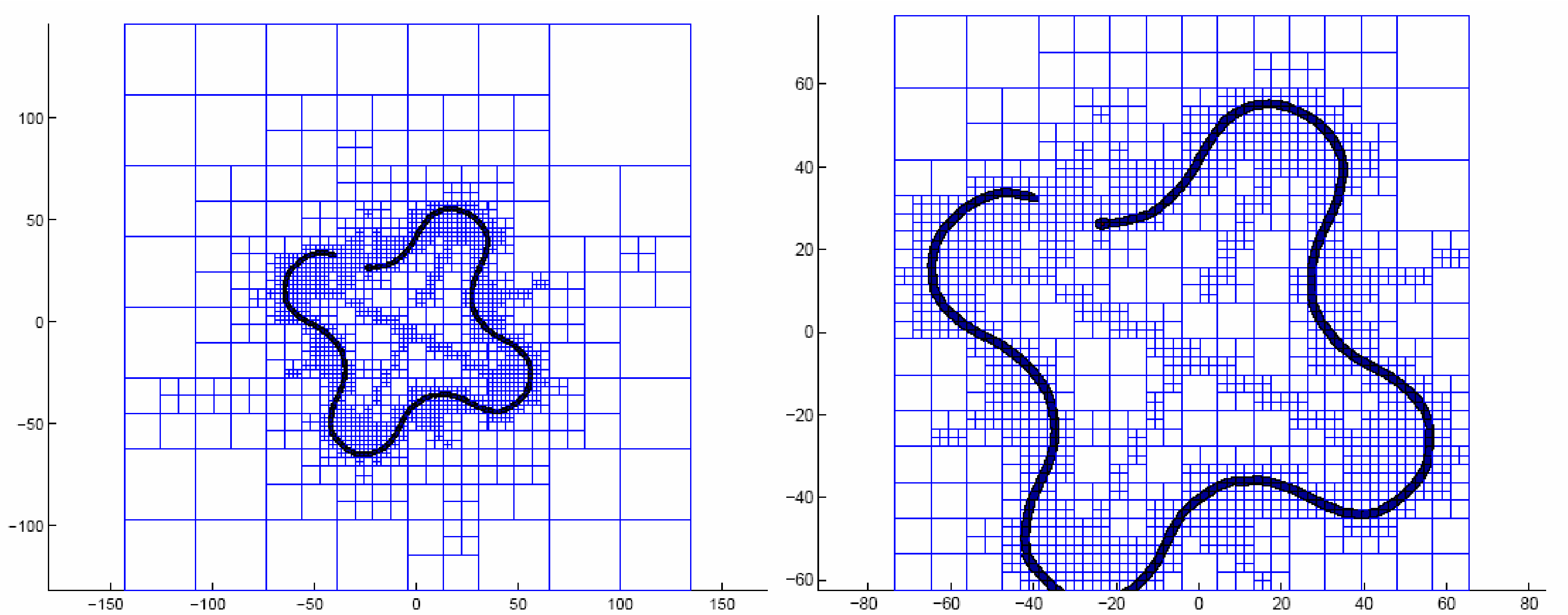
$$f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + c$$

- Start off with the largest cell of the distance-octree, and compute a LS fit using all data points with known distance in the cell.
- If the residual is above a threshold, subdivide the cell, and fit a quadratic to each cell separately.
- Continue in this fashion until an adequate quadratic is obtained for each cell.

3. Computing the d^2-tree

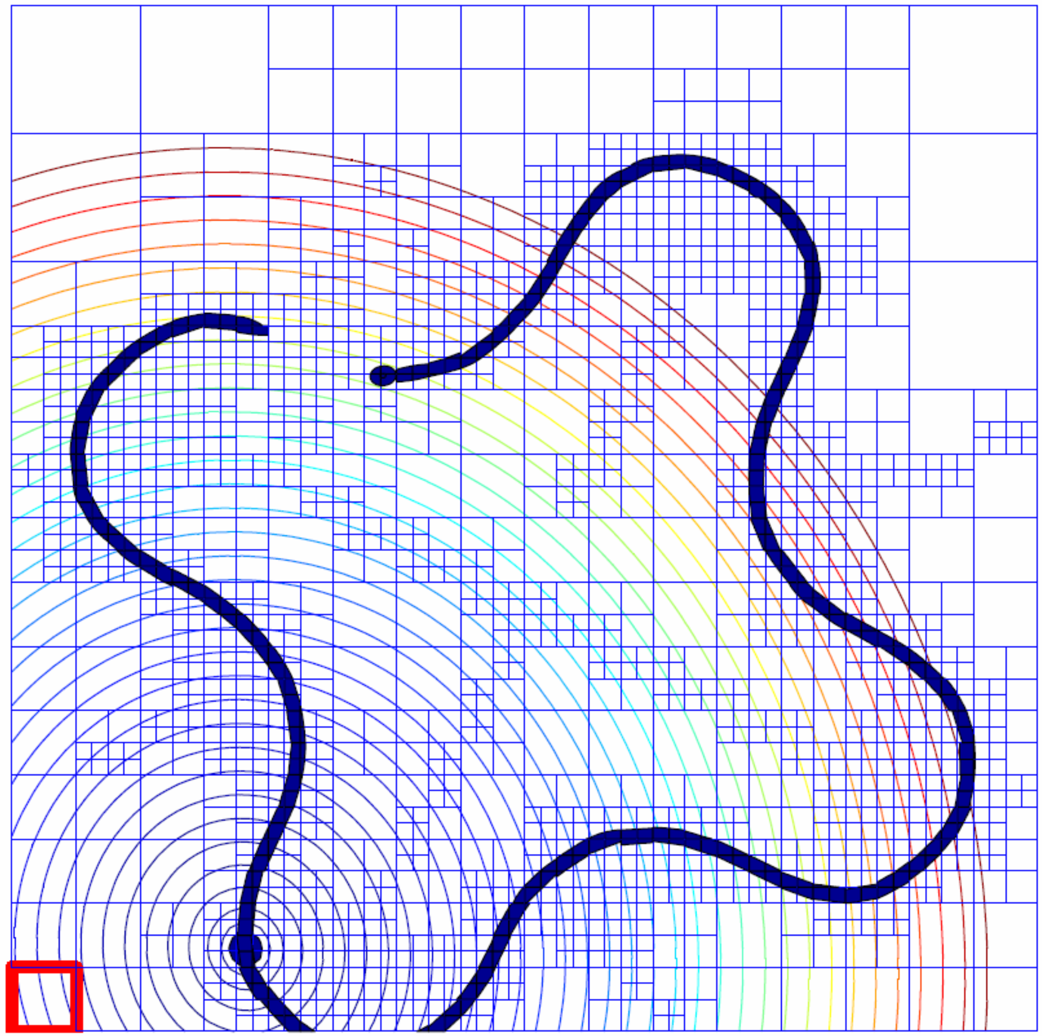
- To avoid excessive laboring in the coarser levels, rather than fitting all points at once, start by fitting only level 2 points, then, if necessary, add level 4 points, and so on.
- We end up with an d^2-tree containing local quadratic approximations of the squared distance function.

d²-tree - Example



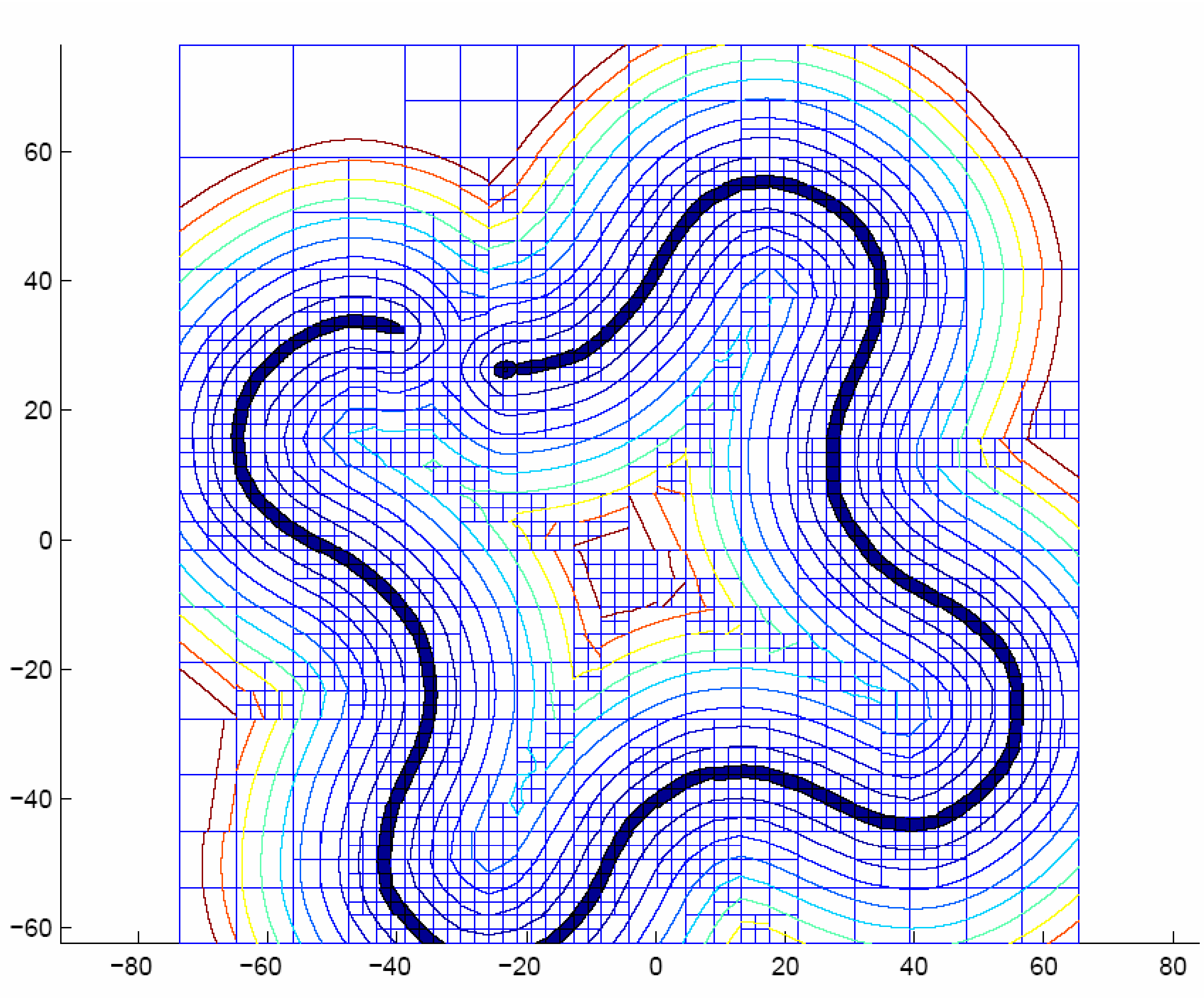
d^2-tree - Example

- A local quadratic approximation with its level sets



d²-tree - Example

- Combined level sets of quadratic approximations



d^2-tree - Example

- Piecewise-quadratic d^2 function