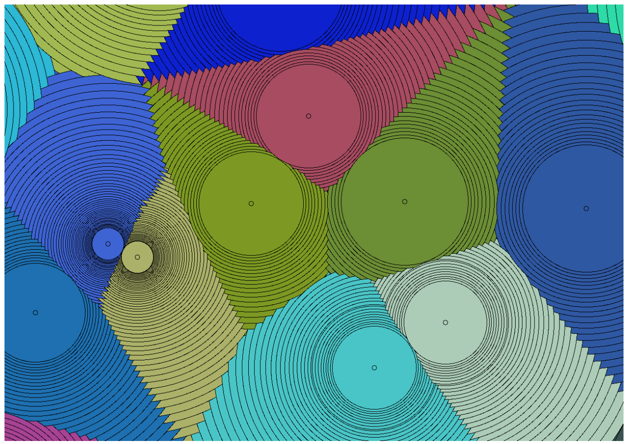
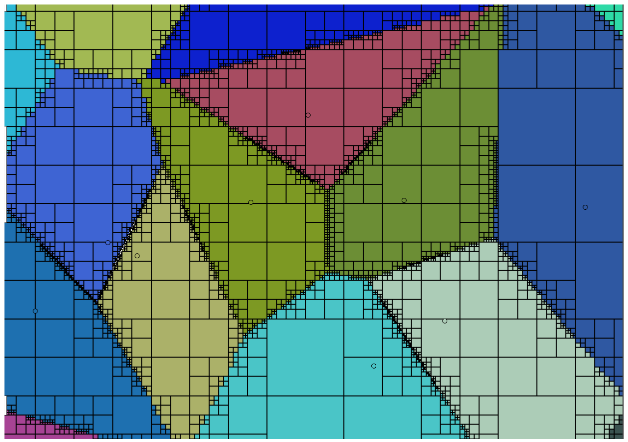
Approximate Voronoi Diagrams

Presentation by Maks Ovsjanikov



S. Har-Peled's notes, **Chapters 6** and **7**



Outline

- Preliminaries
- Problem Statement
- ANN using PLEB
  } (Previous Lecture)
- Bounds and Improvements
  - Near Linear Space
  - Linear Space
- ANN in $\mathbb{R}^d$ using compressed quad-trees

Preliminaries

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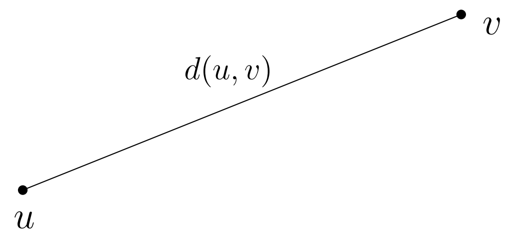

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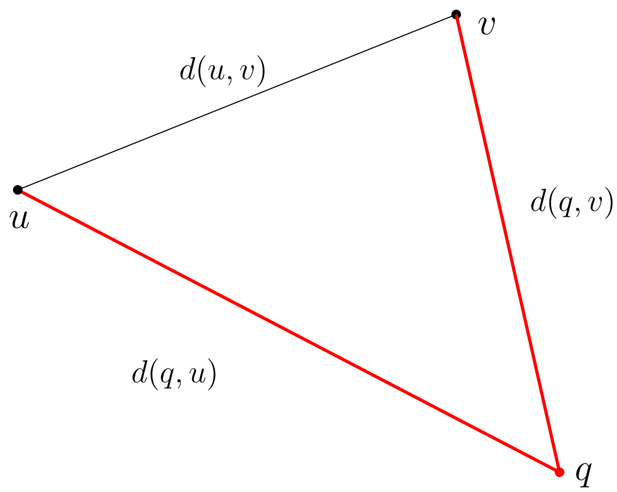

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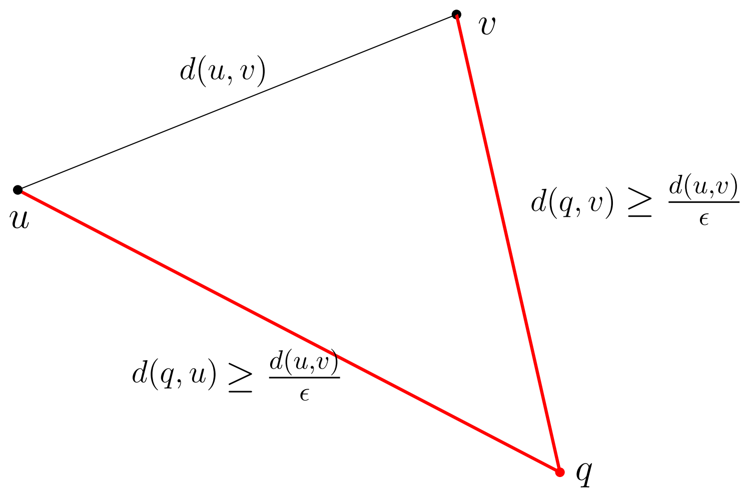

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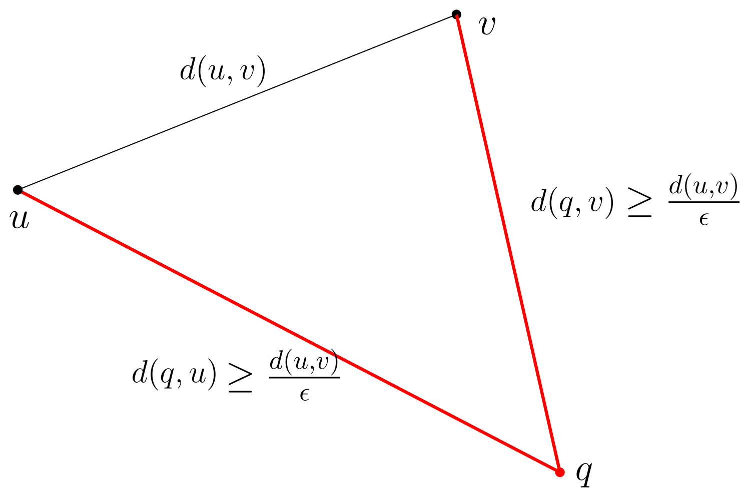

$$\begin{cases} d(q, u) \geq \dfrac{d(u,v)}{\epsilon} \\ d(q, v) \geq \dfrac{d(u,v)}{\epsilon} \end{cases} \Longrightarrow \boxed{\dfrac{d(q, v)}{d(q, u)} \leq 1 + \epsilon}$$

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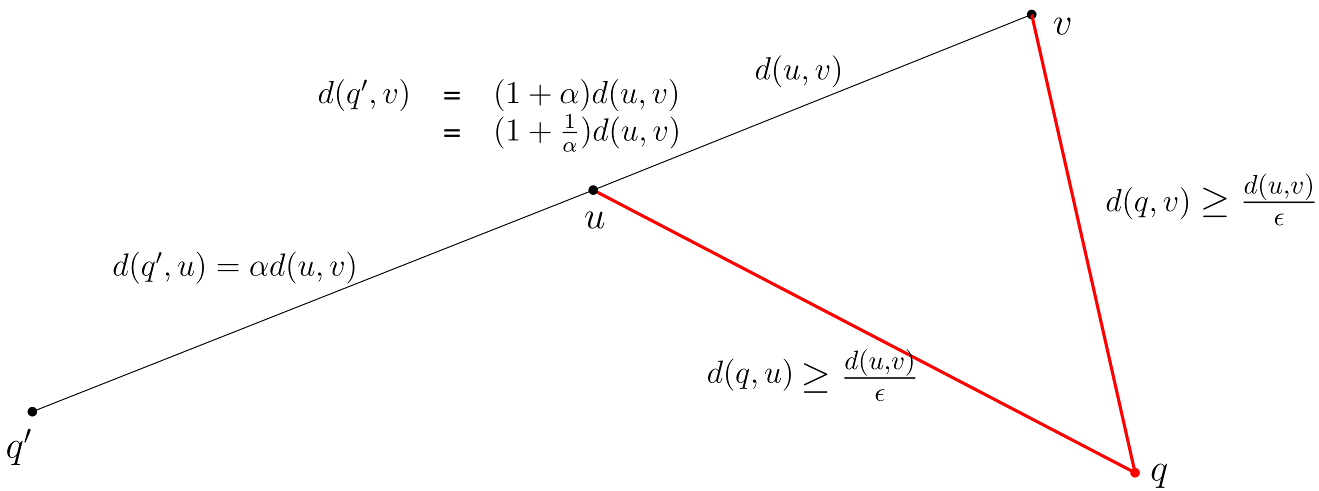

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Holds in any metric space:

$$d(q,u) = \alpha d(u,v)$$

$$d(q,v) \leq d(q,u) + d(u,v) = (1+\tfrac{1}{\alpha})d(q,u)$$

$\Longrightarrow \frac{d(q,v)}{d(q,u)} \leq (1+\frac{1}{\alpha}) \leq (1+\epsilon)$ if $\alpha \geq \frac{1}{\epsilon}$

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Similarly:

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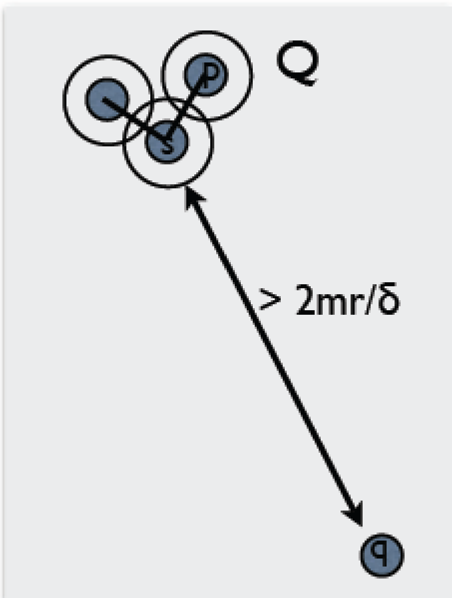
Moral:

Any of the far away points is a $(1+\epsilon)$ closest neighbor



Problem Statement:

For a given ϵ, find a $(1+\epsilon)$ Aproximate Voronoi Diagram:

Partition of space into regions with one representative r_i per region, such that for any point q in region i, r_i is a $(1+\epsilon)$ nearest neighbor of q

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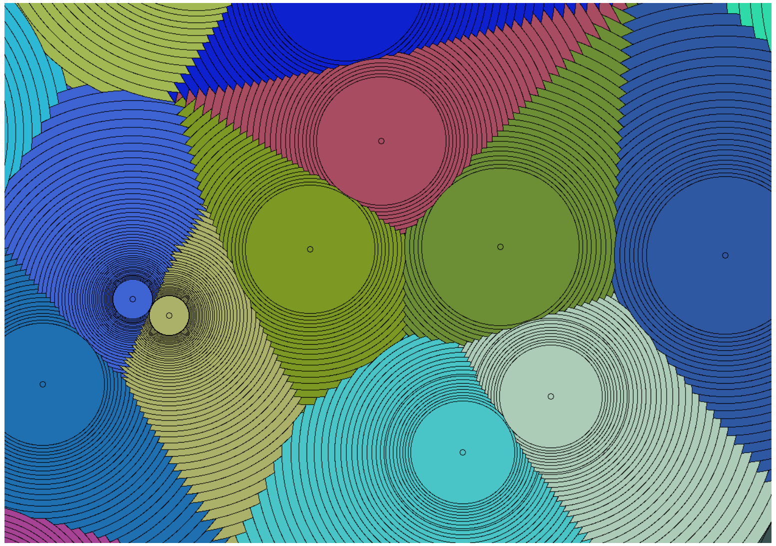

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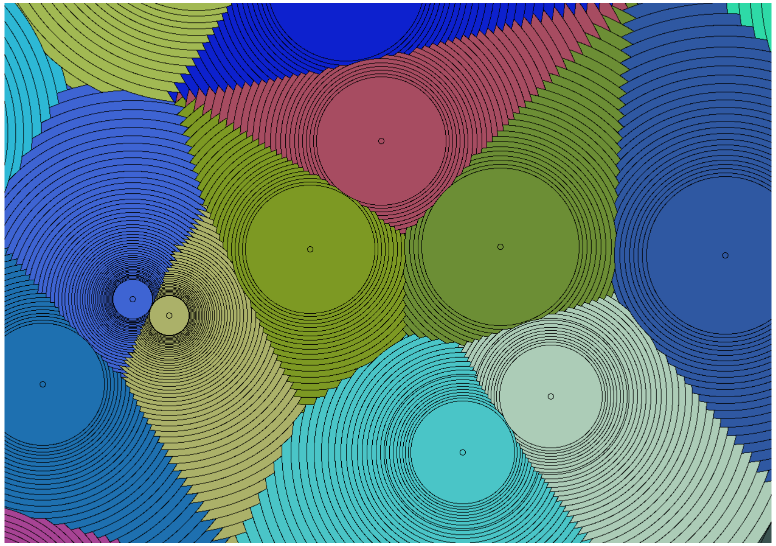

Constraints:

- bounded construction time and space (complexity)
- Cover all space
- sub-linear (1+ϵ) NN queries

ANN using PLEB

Reduce $(1+\epsilon)$-ANN queries to target ball queries

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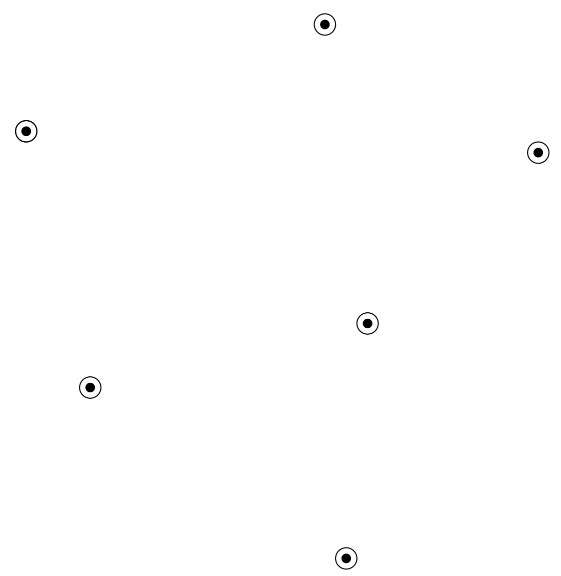

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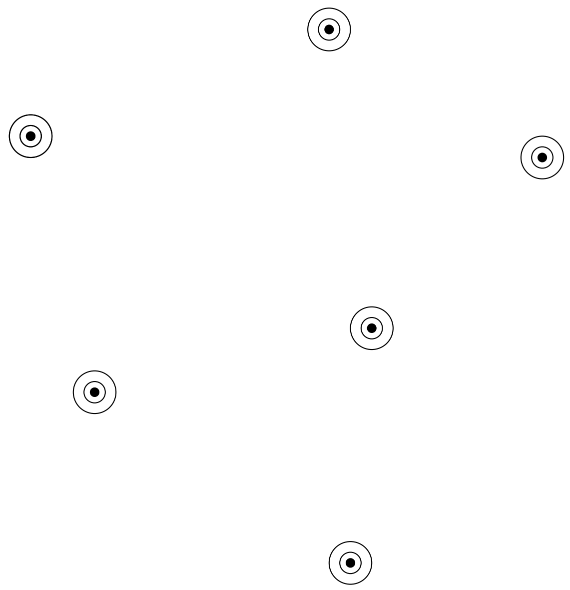

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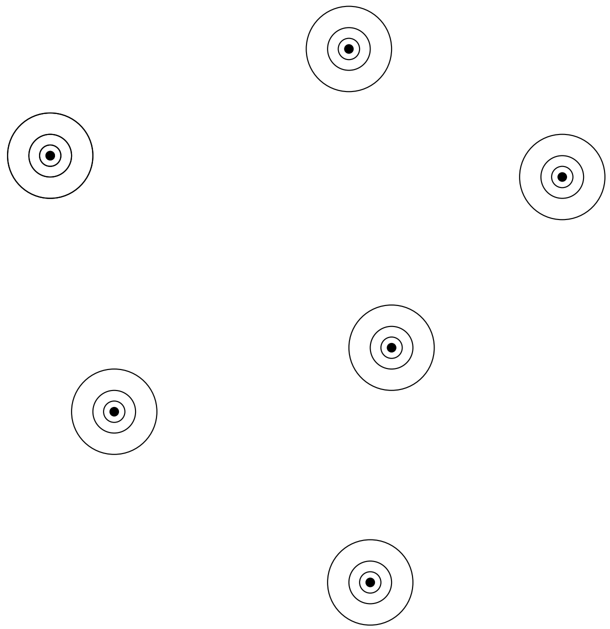

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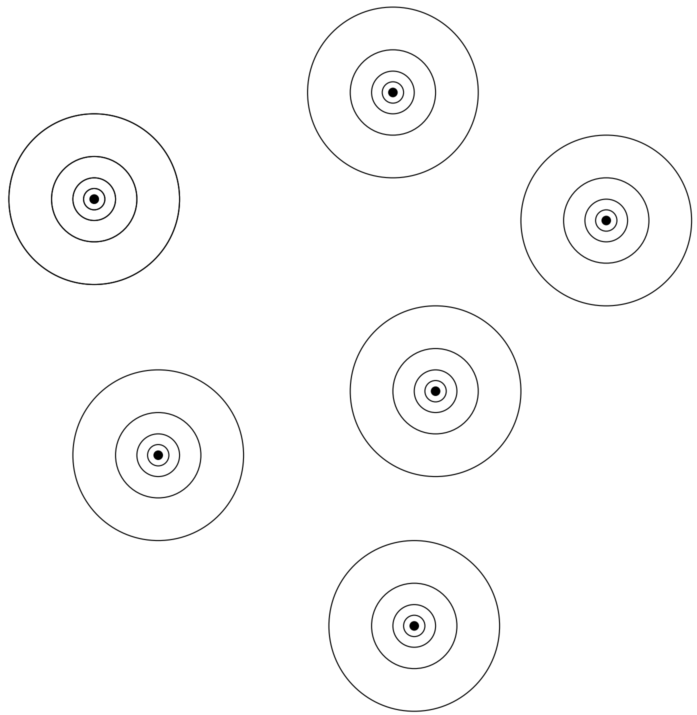

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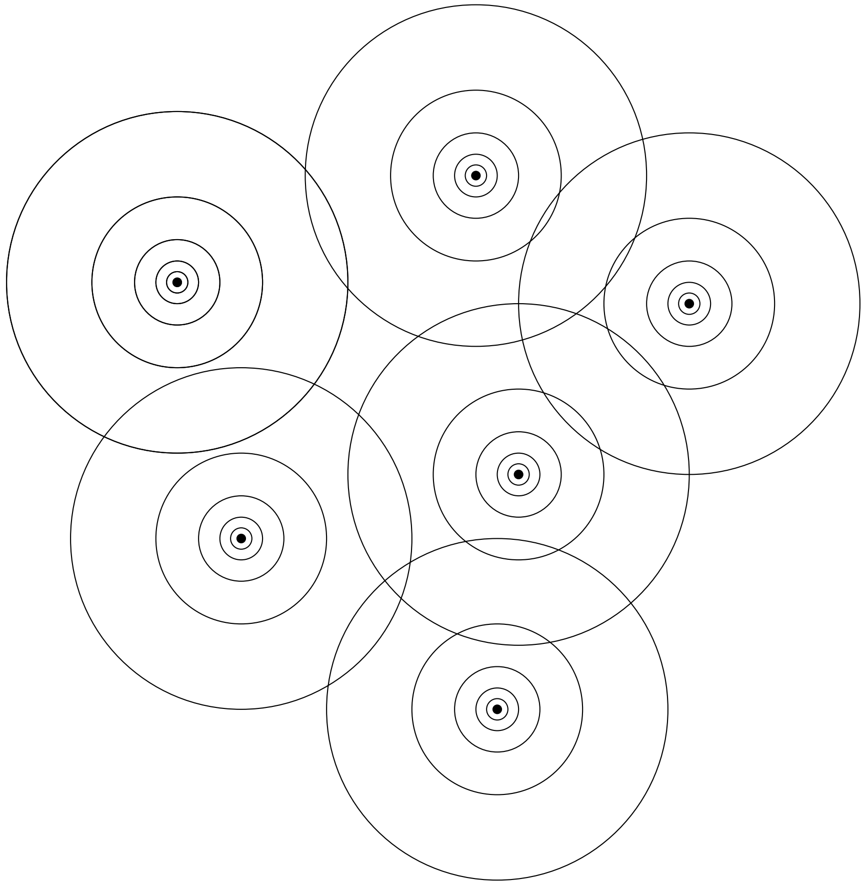

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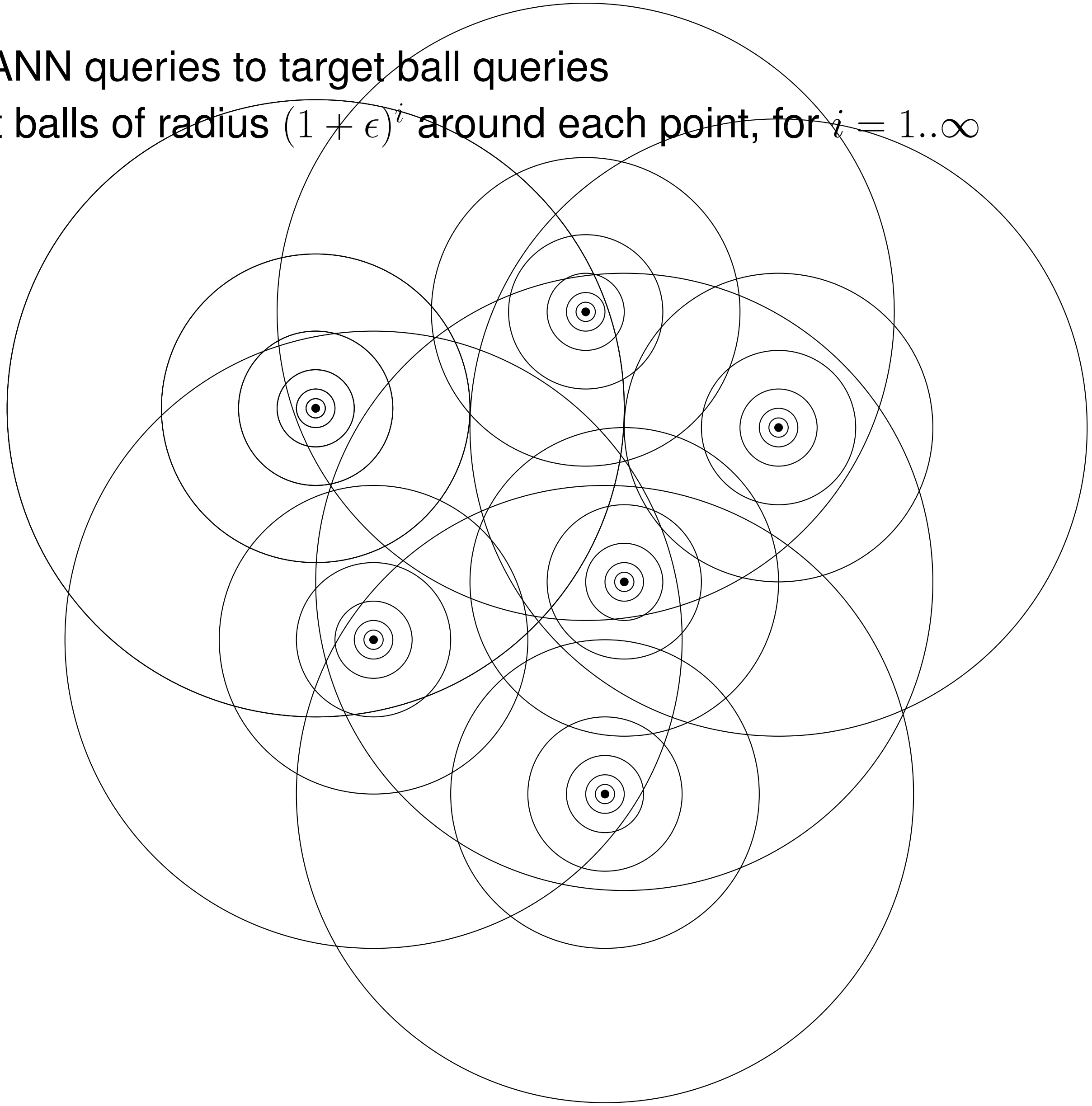

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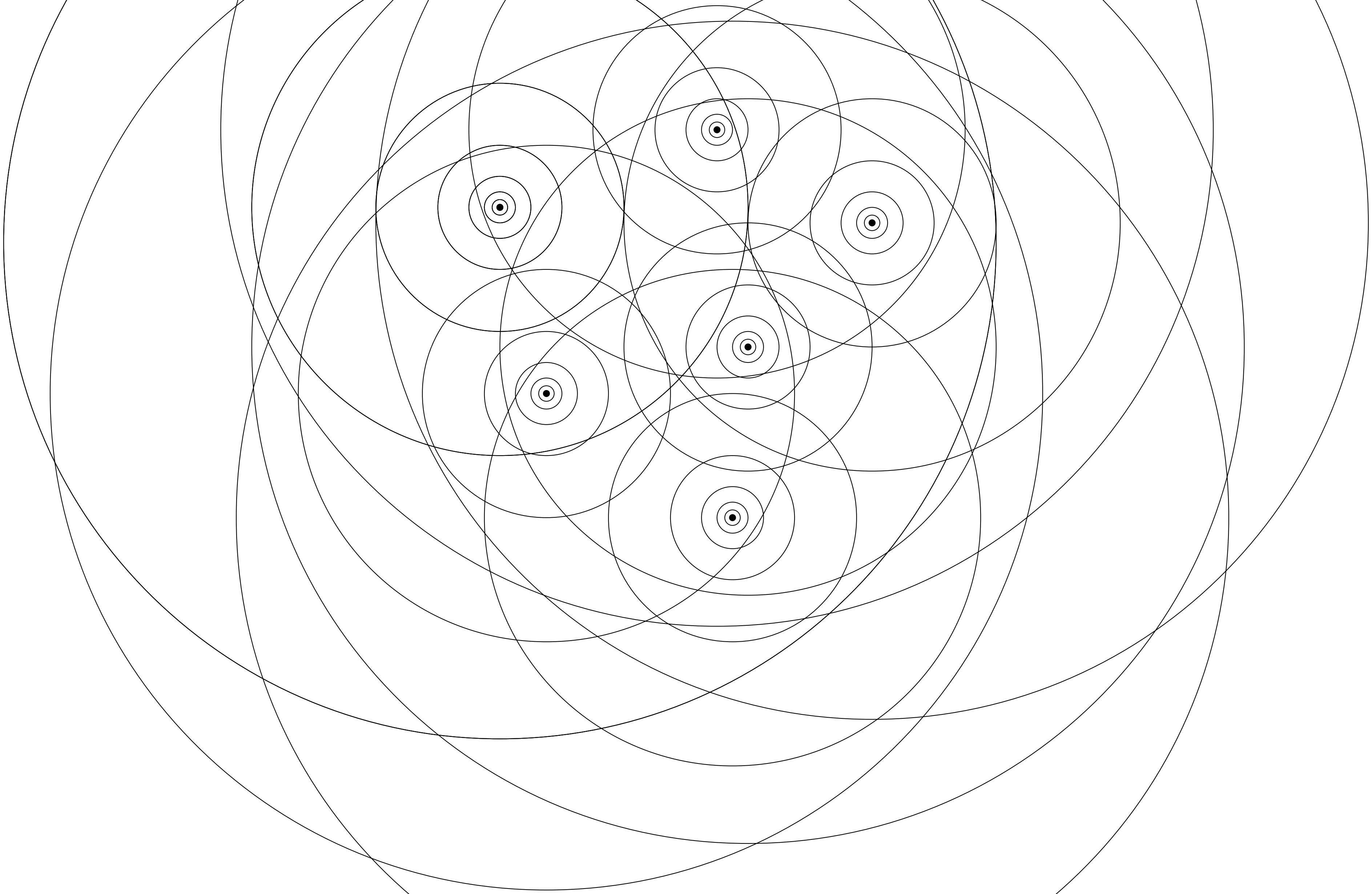

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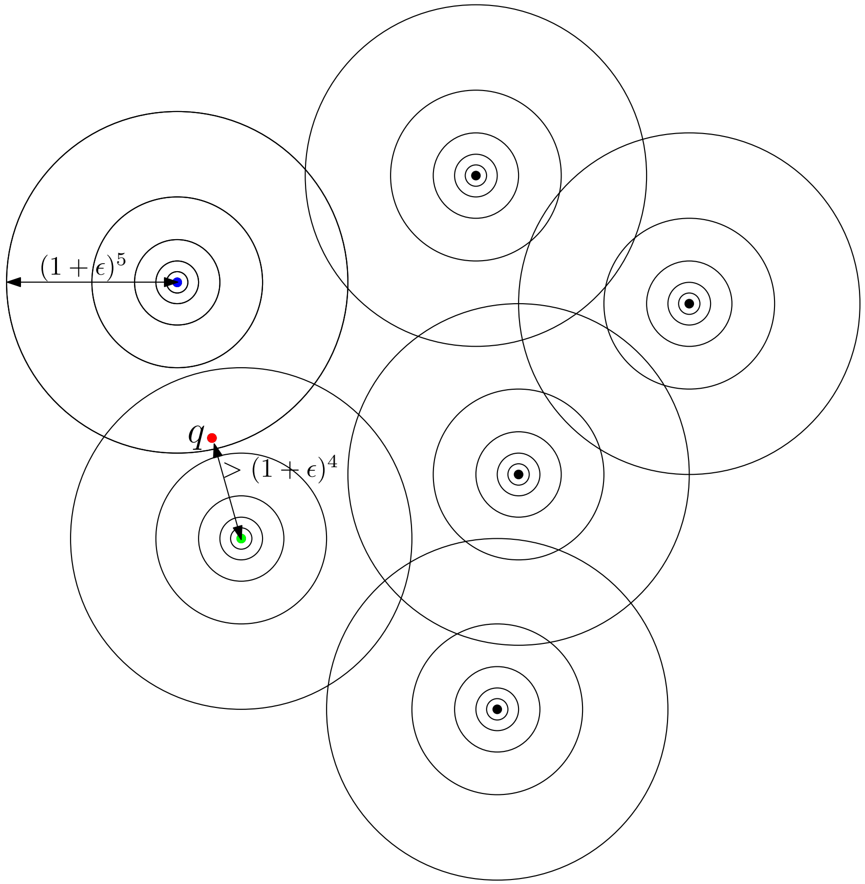

For any query point q, return the center p of the smallest ball that contains it:

$d(q,n) > (1+\epsilon)^{i-1}$, and $d(q,p) \leq (1+\epsilon)^i < (1+\epsilon) \cdot d(q,n)$

$\Longrightarrow$ always get a $(1+\epsilon)$-Nearest Neighbor

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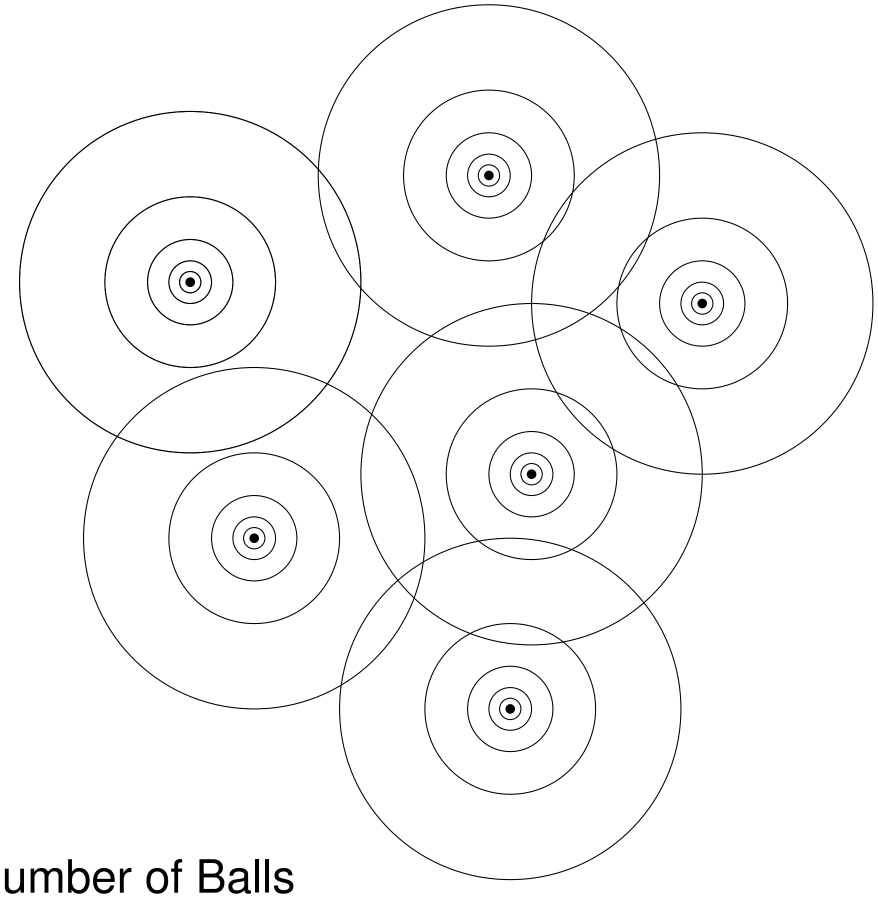

Problems:

- Unbounded Number of Balls
- Not clear how to preform target ball queries efficiently
  - Partition the space into regions of influence

Bounding the number of balls

Intuition:

* For a given pair u and v, we only care if $\min d(q, \{u, v\}) \in [\frac{d(u,v)}{\epsilon+2}, \frac{d(u,v)}{\epsilon}]$

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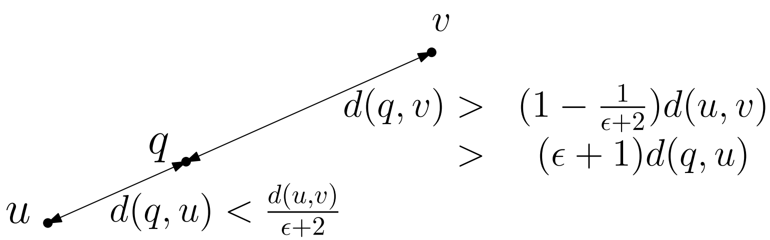

* Do not need to grow balls of radius smaller than $\frac{d(u,v)}{4}$ or larger than $\frac{2d(u,v)}{\epsilon}$

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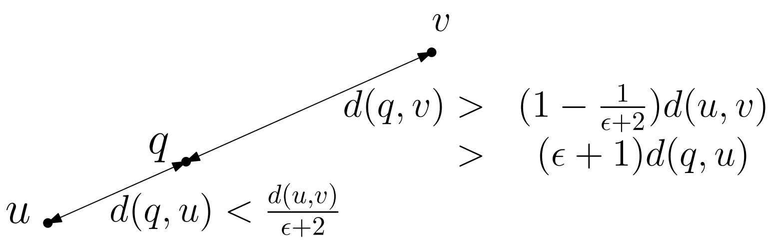

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Method 1:

for every pair of points $\{u, v\}$, construct enough balls to cover $[\frac{d(u,v)}{4}, \frac{2d(u,v)}{\epsilon}]$ on u, v

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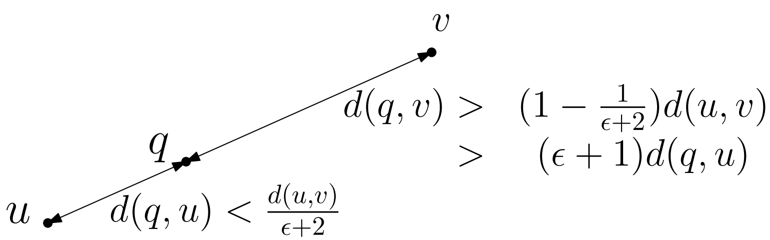

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Overall: $O(n^2 \log_{\epsilon+1}(\frac{2C}{\epsilon} - \frac{C}{4})) = O(n^2 \frac{\log(\frac{7C}{\epsilon})}{\log(\epsilon+1)}) = O(n^2 \frac{1}{\epsilon}\log(\frac{1}{\epsilon}))$ balls

Note: $\log(1+\epsilon) = \epsilon - \epsilon^2/2 + \epsilon^3/3 - .... = O(\epsilon)$ in most cases

Bounding the number of balls

Interval Near-Neighbor data structure

given a range of distances $[a, b]$, and a set of points P, answers:

1. $d_P(q) > b$
2. $d_P(q) < a$ with a witness
3. otherwise, finds a point $p \in P$, s.t. $d_P(q) \leq d(p, q) \leq (1 + \epsilon)d_P(q)$

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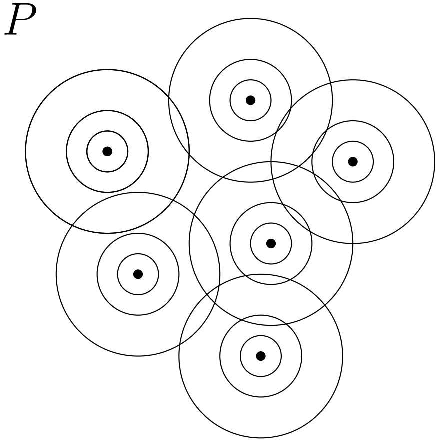
Can be realized by a set of balls of radius $a(1 + \epsilon)^i$ for $i = 0...M - 1$, where $M = \lceil \log_{1+\epsilon}(b/a) \rceil$ and a ball of radius b around every point in P



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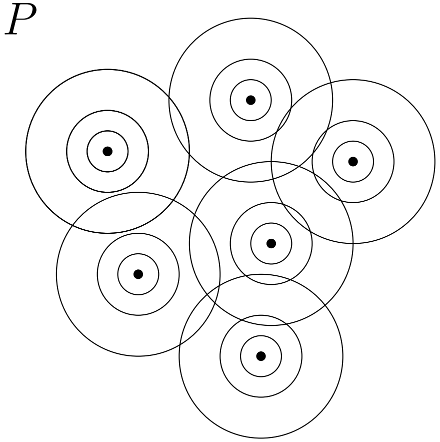

Contains $O(n\frac{1}{\epsilon}\log(b/a))$ balls. Takes at most 2 target ball queries if 1 or 2 hold, and

* $O(\log(M)) = O(\log \frac{\log(b/a)}{\epsilon})$ otherwise

Bounding the number of balls

A data structure to answer $(1+\epsilon)$-ANN queries on general points

Build a tree, with an Interval Near Neighbor structure associated with each node

(Sariel Har-Peled: *A Replacement for Voronoi Diagrams of Near Linear Size*. FOCS 2001: 94-103)

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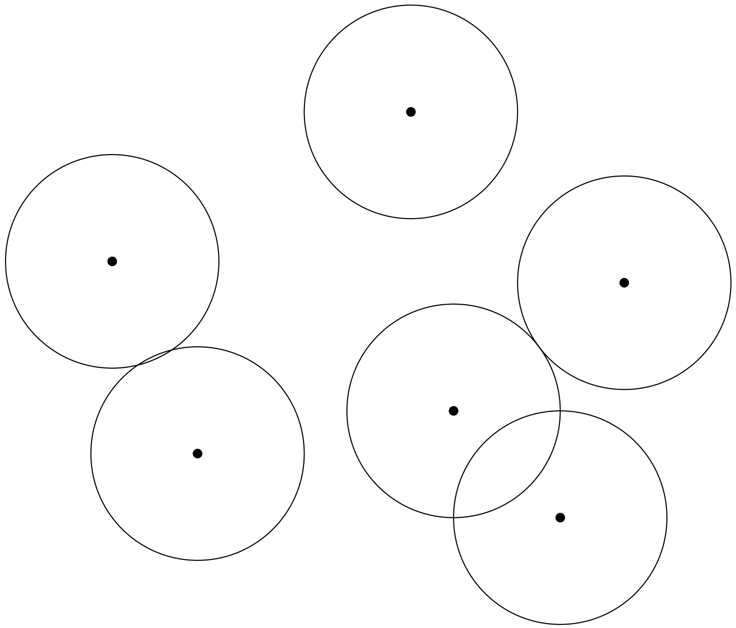

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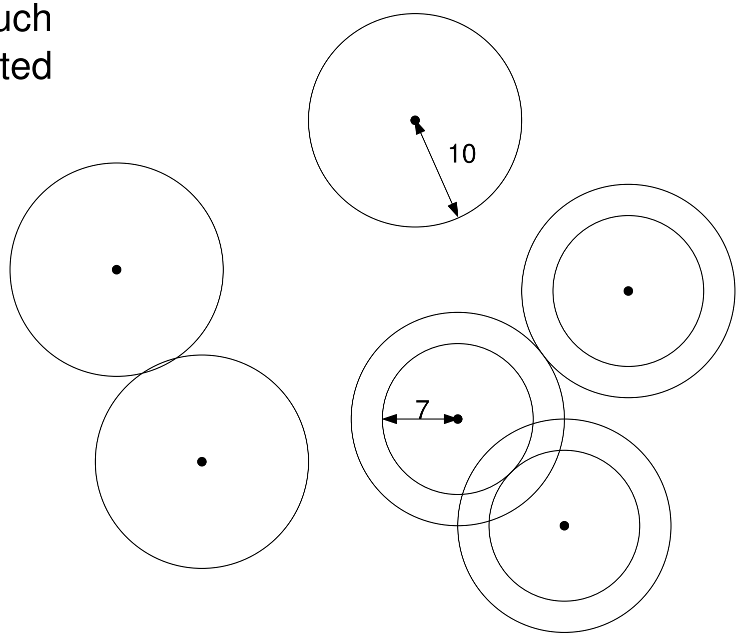
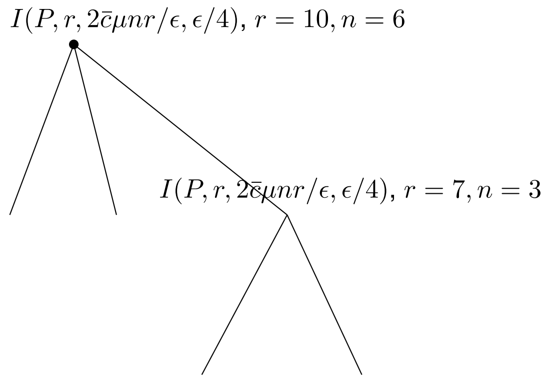
Recursively find $\min r$ such that there are $\lceil n/2 \rceil$ connected components





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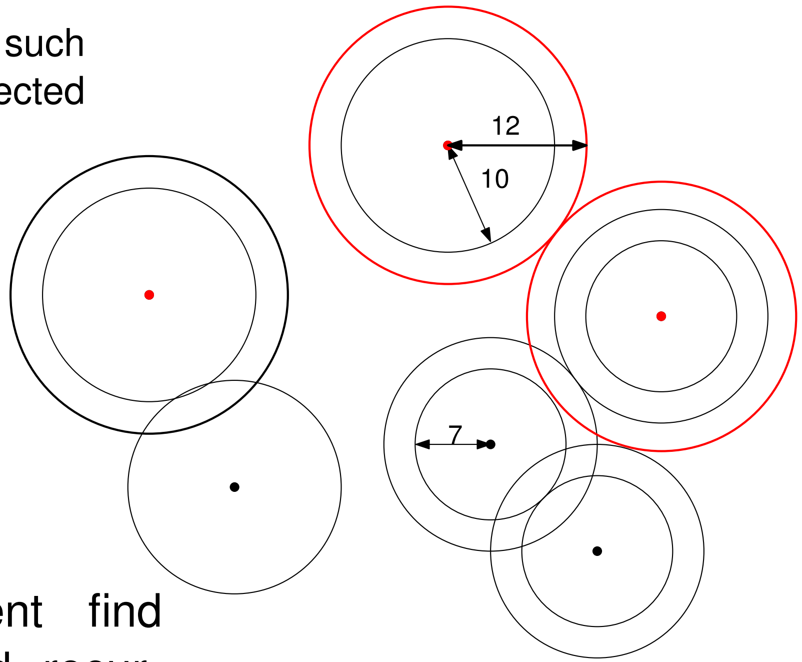

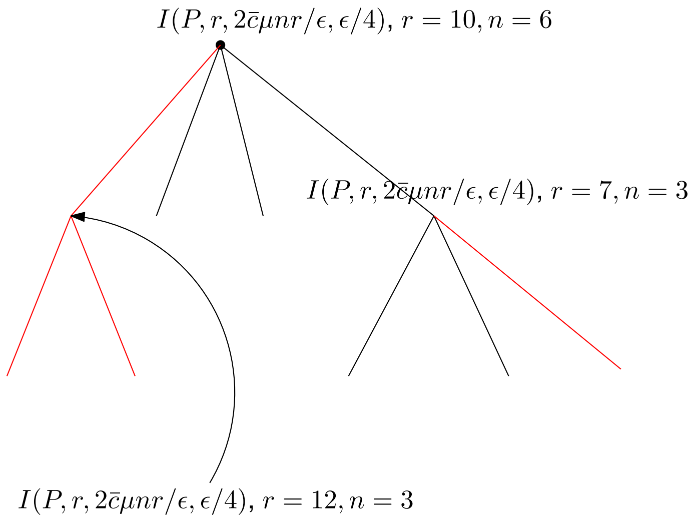

For each component find a representative and recursively build the outer tree

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Bounding the number of balls

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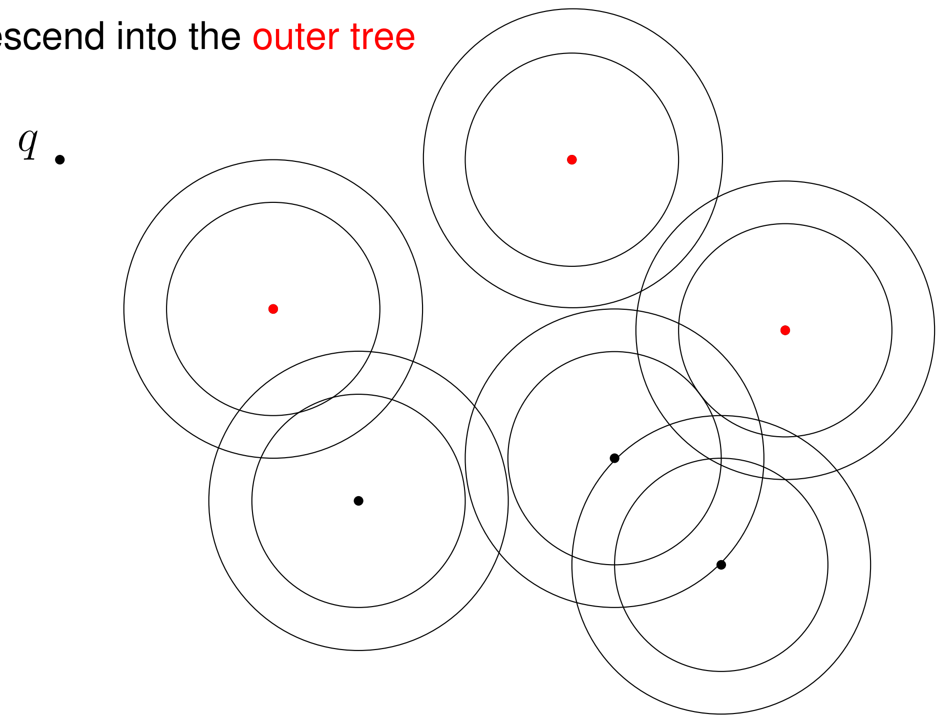
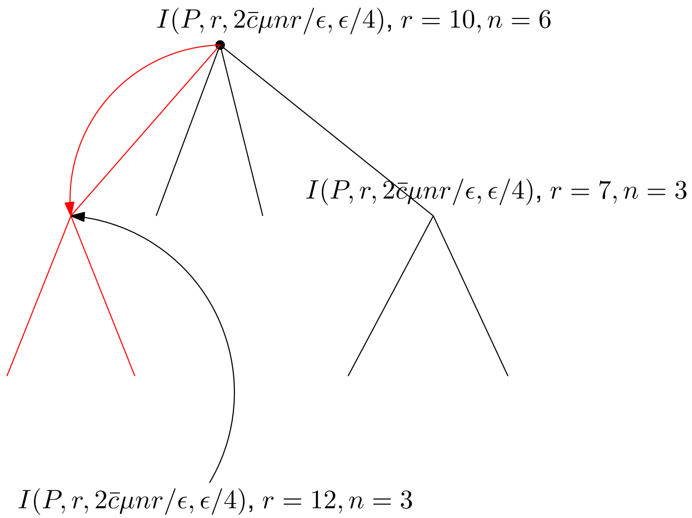
Given a query point q:

1) q is outside R descend into the outer tree





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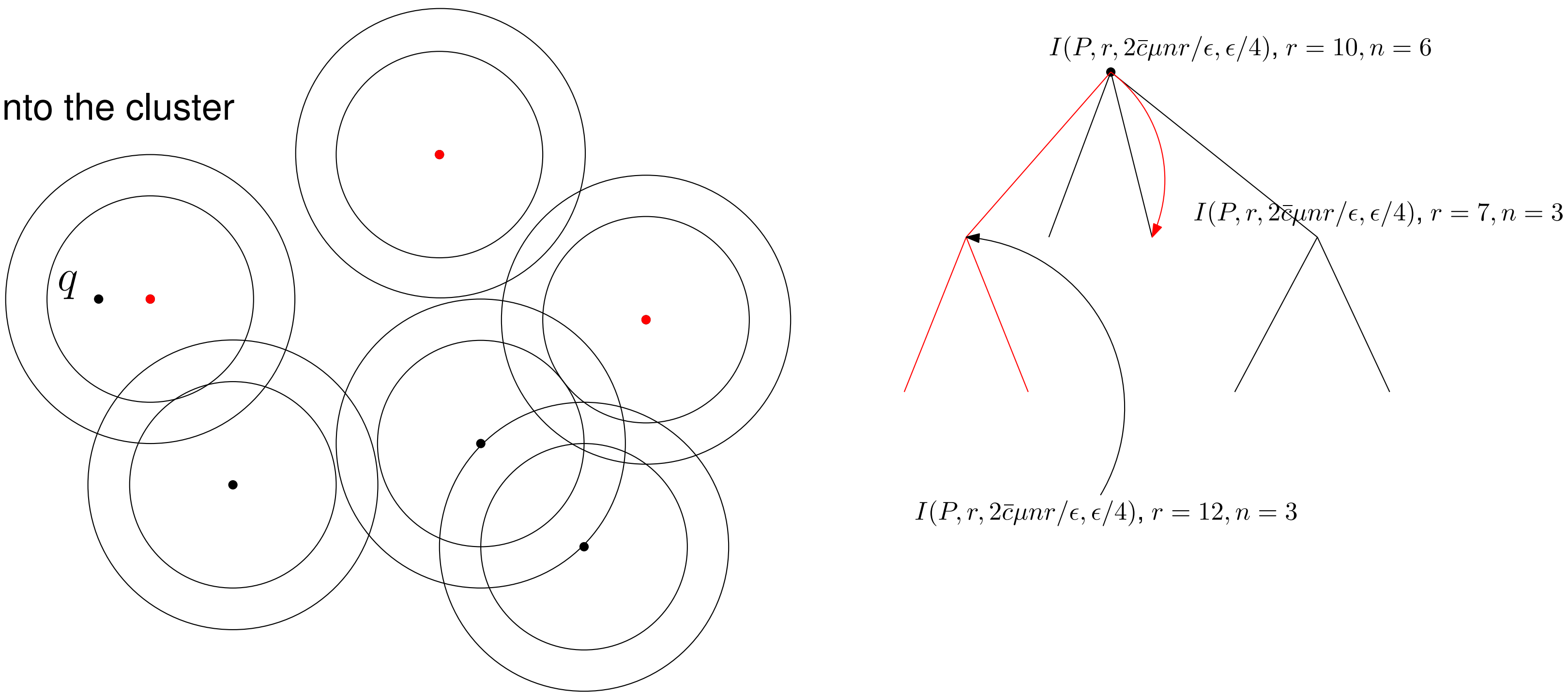
Bounding the number of balls

A data structure to answer $(1+\epsilon)$-ANN queries on general points

Given a query point q:

2) if q is inside r descend into the cluster



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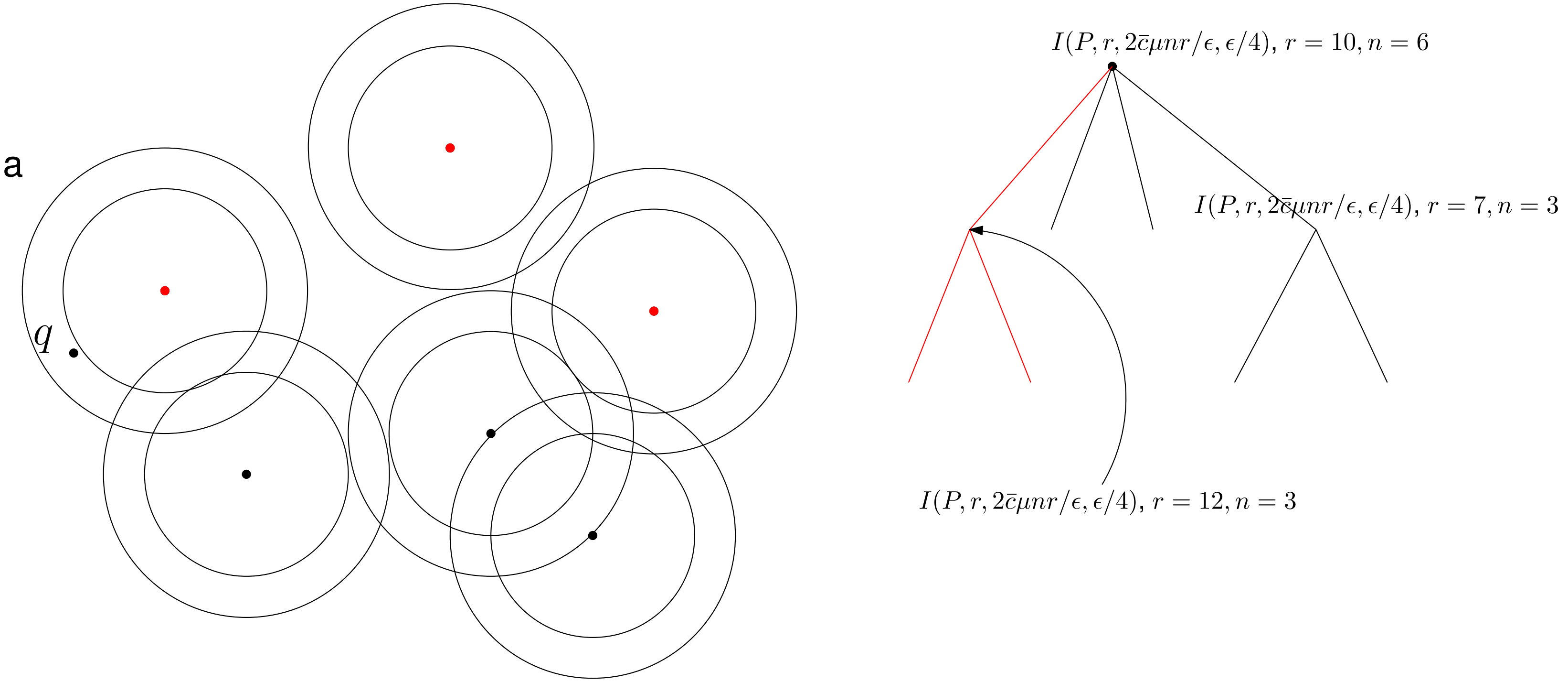
Bounding the number of balls

A data structure to answer $(1+\epsilon)$-ANN queries on general points

Given a query point q:

3) otherwise I will return a $(1+\frac{\epsilon}{4})$-NN



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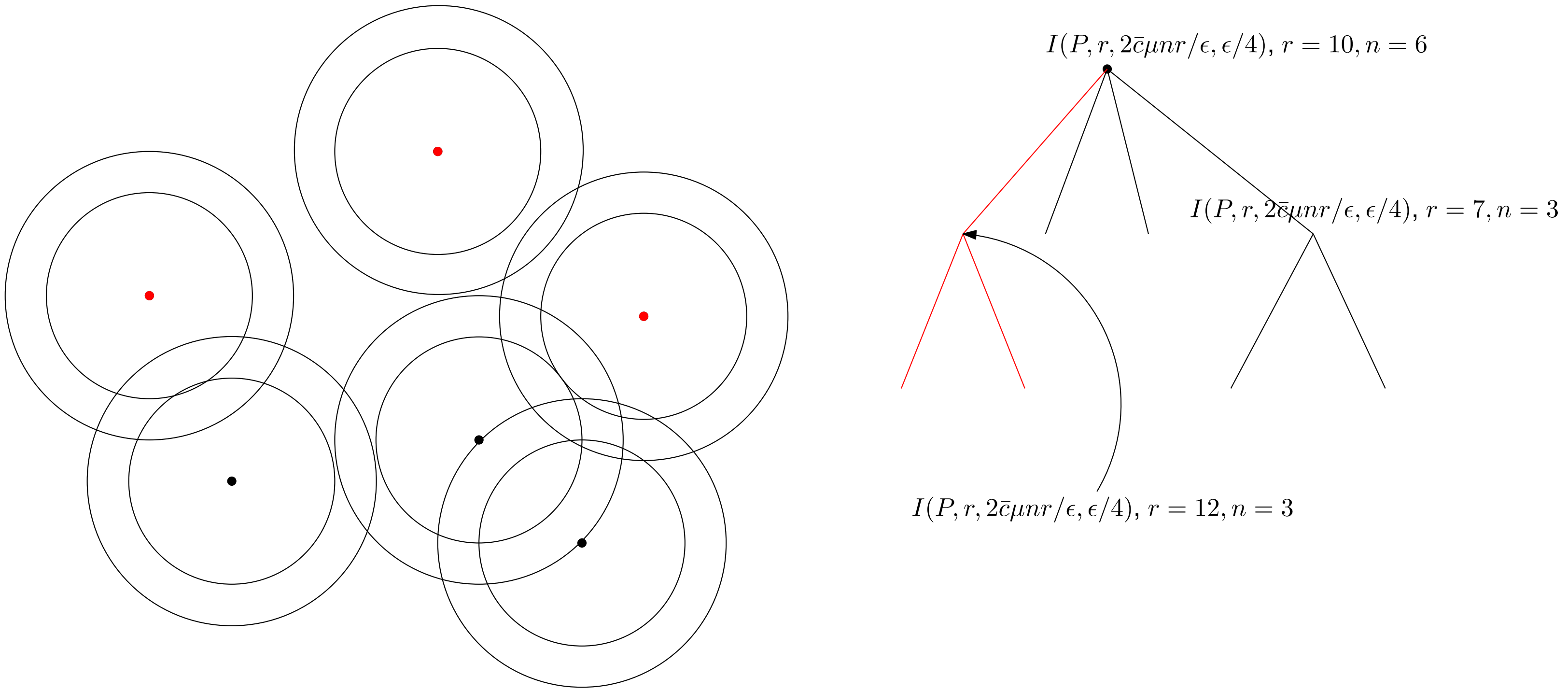

Because of rounding up, after each step, continue on set containing $\leq n/2 + 1$ points
$\Longrightarrow$ number of steps $\leq \log_{3/2} n$

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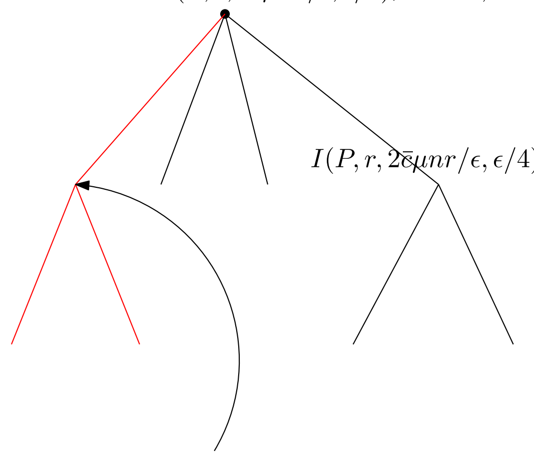


Note that:

- last step is always 3)
- no error is incurred in 2)
- diameter of a cluster $\leq 2nr \implies$ error in 1) is at most $(1+\frac{\epsilon}{\bar{c}\mu})$

Thus, overall error is bounded by:

$$(1+\frac{\epsilon}{4})\prod_{i=1}^{\log_{3/2} n}(1+\frac{\epsilon}{\bar{c}\mu}) \leq \exp(\frac{\epsilon}{4})\prod_{i=1}^{\log_{3/2} n}\exp(\frac{\epsilon}{\bar{c}\mu}) \leq \exp\left(\frac{\epsilon}{4}+\sum_{i=1}^{\log_{3/2} n}\frac{\epsilon}{\bar{c}\mu}\right) \leq \exp(\epsilon/2) \leq (1+\epsilon)$$

if $\mu = \lceil \log_{3/2} n \rceil$, $\bar{c} = 4$ and $\epsilon < 1$

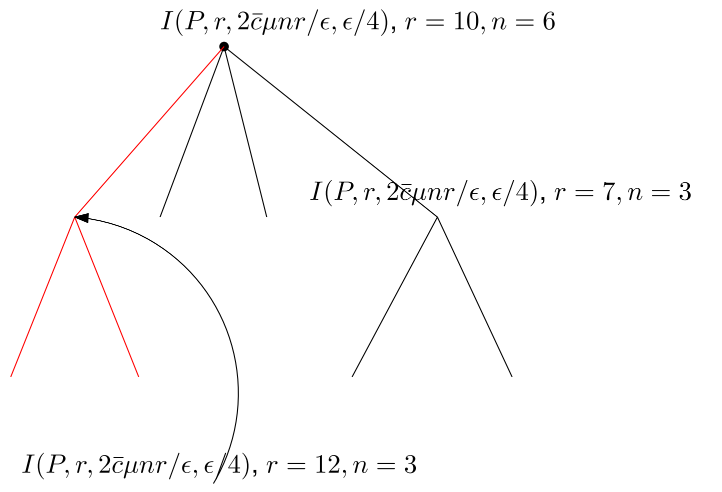
Bounding the number of balls

Overall Number of Balls:

Since

- the depth of the tree is at most $\log_{3/2} n$
- each node ν has $I(P_\nu, r, 2\bar{c}\mu nr/\epsilon, \epsilon/4)$ with $M = n \log n$ balls

we get an immediate bound of
$O(M \log M) = O(n \log(n) \log(n \log n)) = O(n \log^2 n)$



(Sariel Har-Peled: *A Replacement for Voronoi Diagrams of Near Linear Size*. FOCS 2001: 94-103)

Bounding the number of balls

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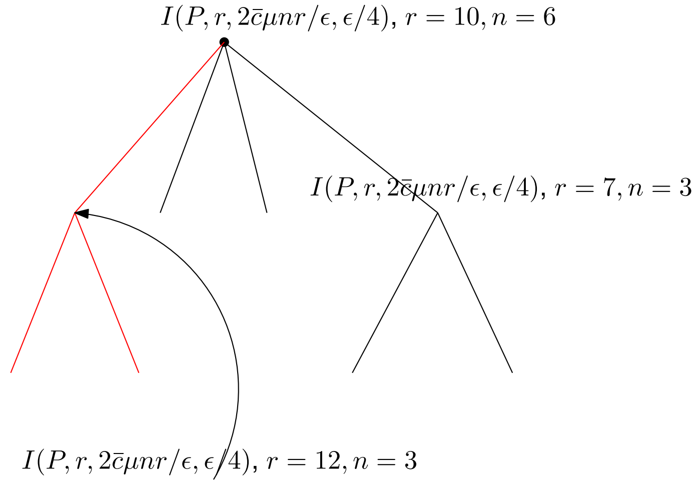

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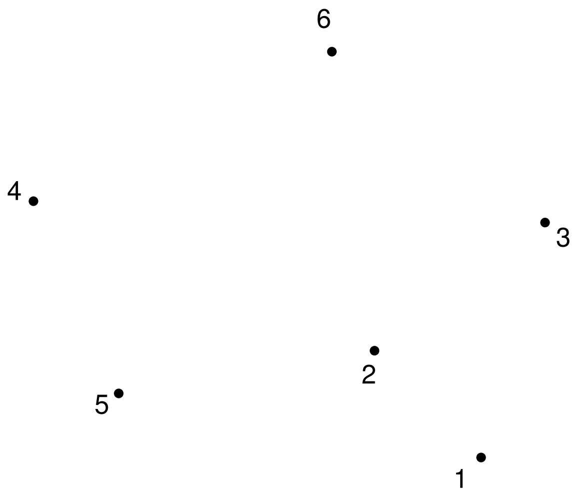

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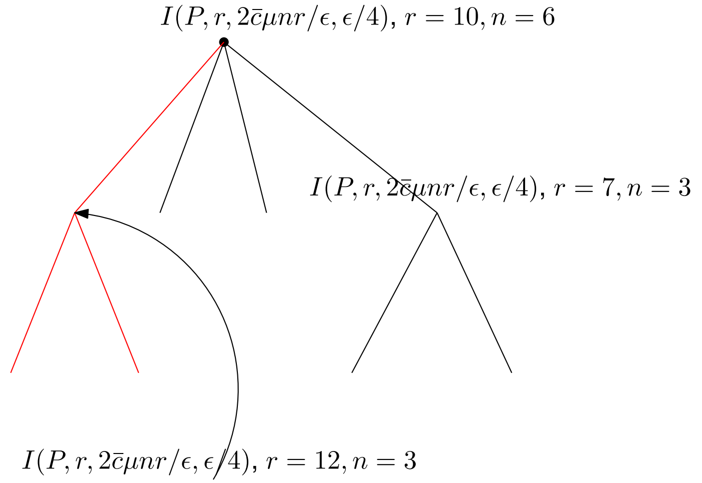

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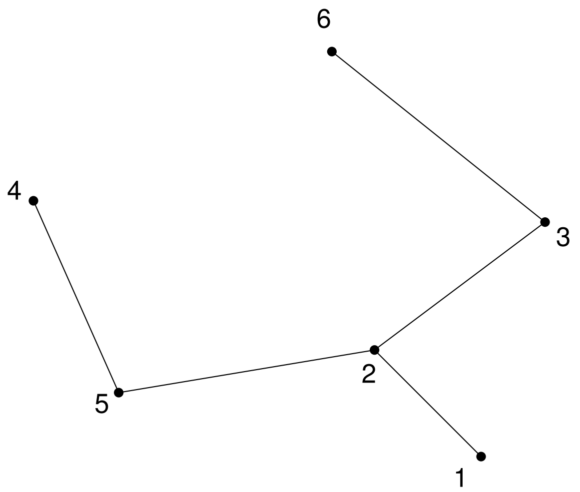

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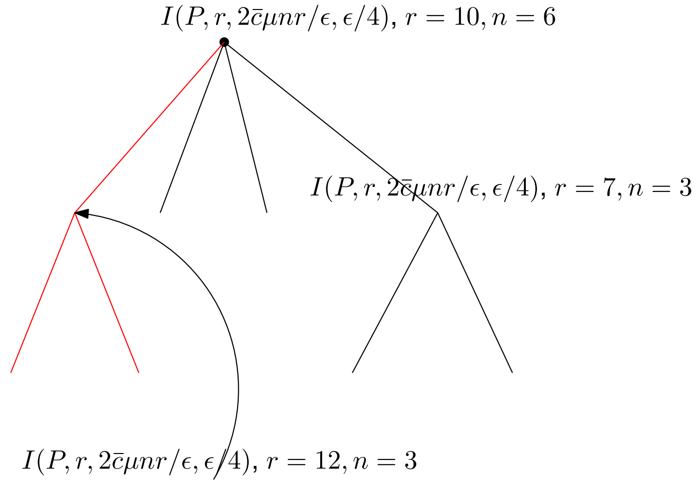

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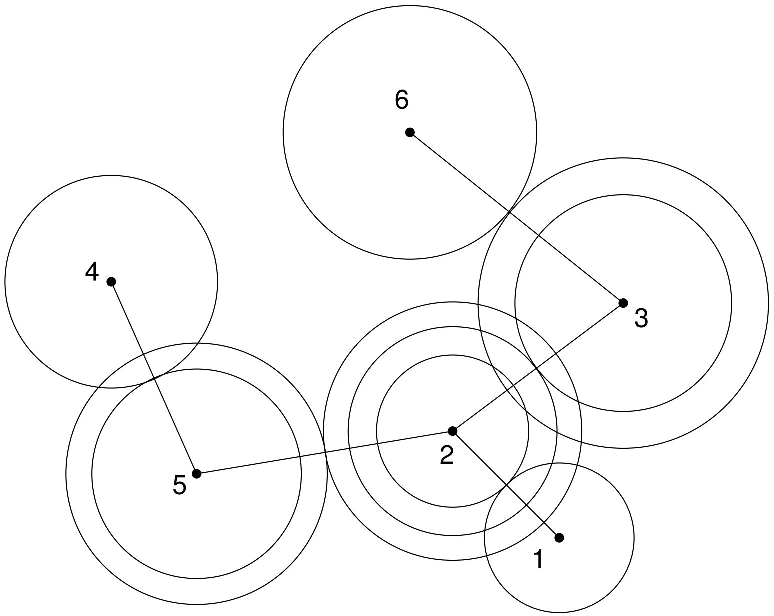

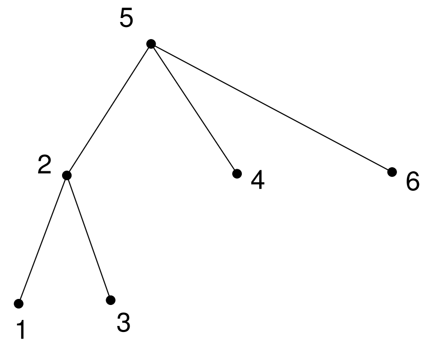

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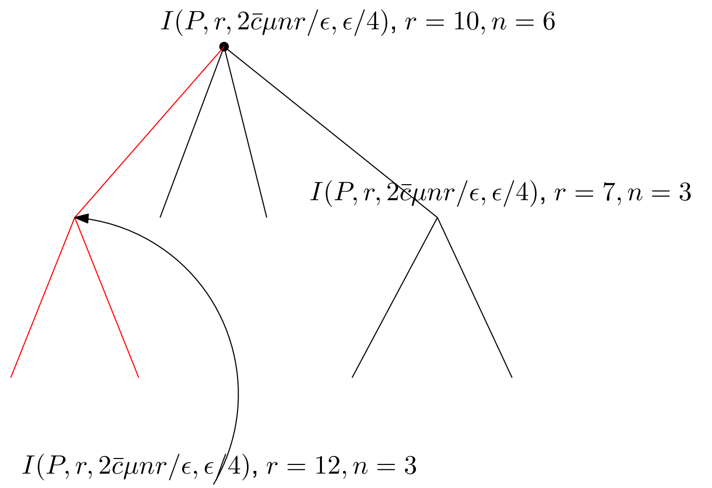

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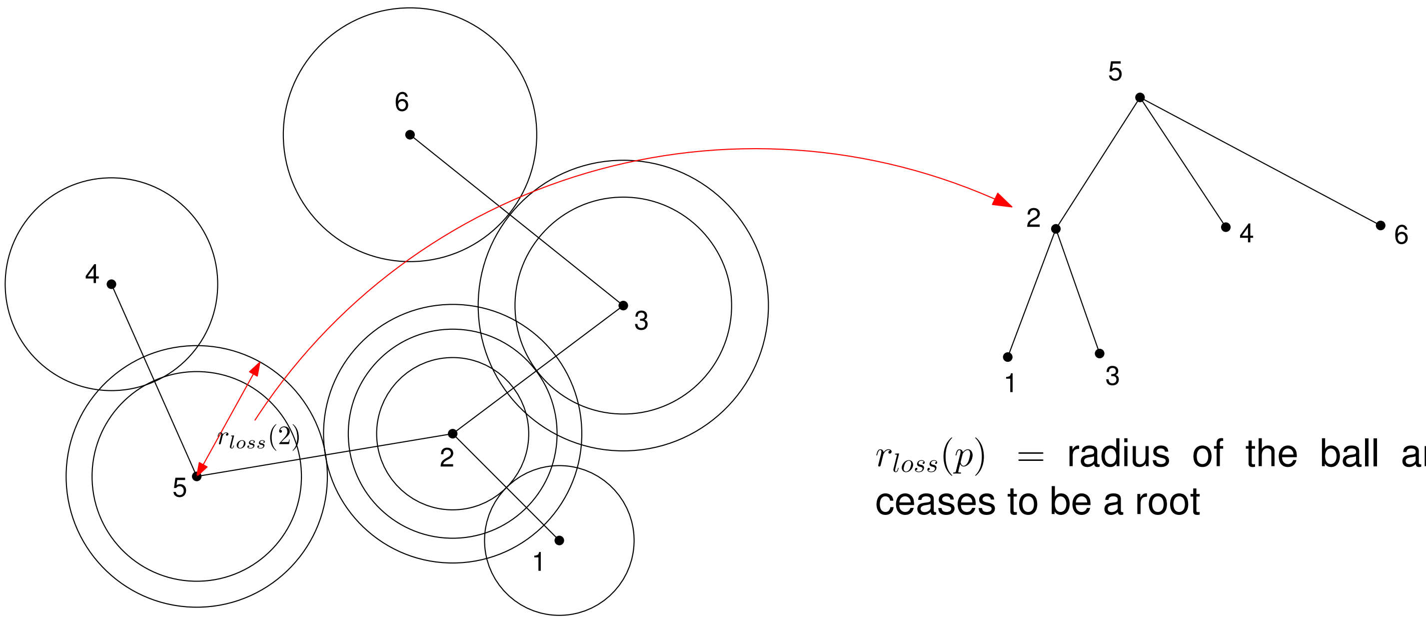
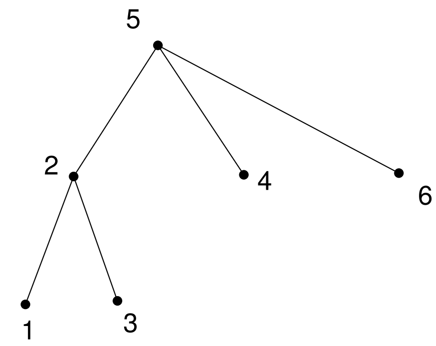
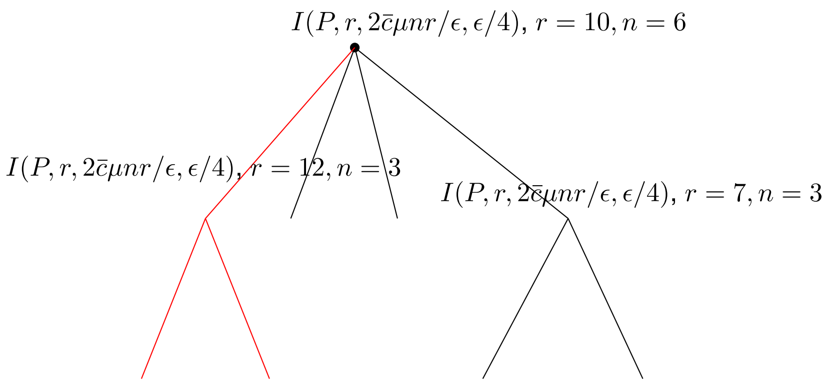


$r_{loss}(p)$ = radius of the ball around p, when p ceases to be a root

Bounding the number of balls

Apart from the outer trees, going down the $(1+\epsilon)$ ANN tree is equivalent to disconnecting edges of the MST tree





The subtrees of a node are disjoint in edges

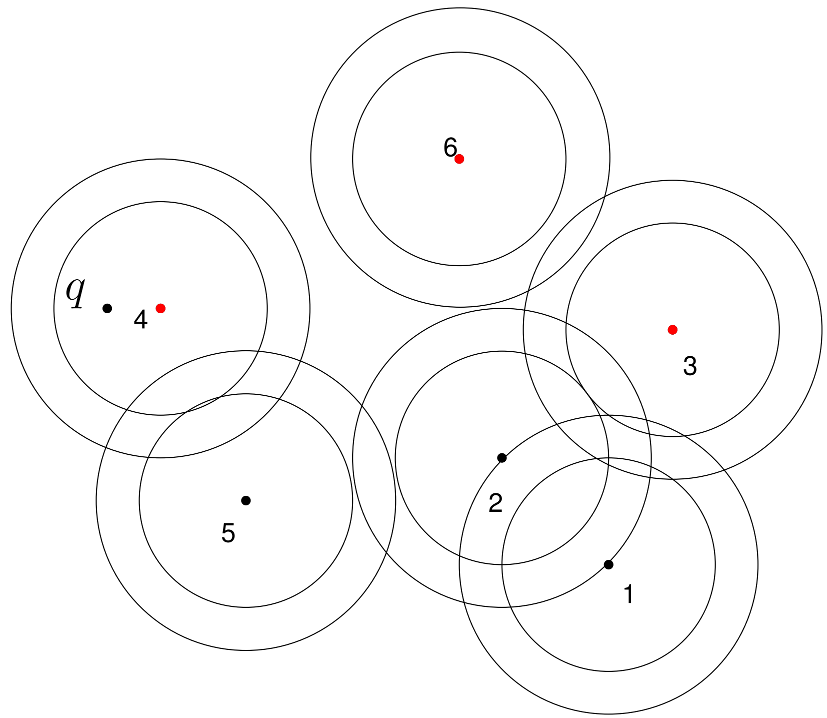
$\Longrightarrow$ can charge at least 1 edge to each child.

Namely: if n_ν is the number of children of ν

$|P_\nu| = O(n_\nu)$ and $\sum_{\nu \in D} n_\nu = O(n)$

Thus, total number of balls:

$$\sum_{\nu \in D} O\left(\frac{n_\nu}{\epsilon} \log \frac{\mu n_\nu}{\epsilon}\right) = O\left(\frac{n}{\epsilon} \log \frac{n \log n}{\epsilon}\right) = O\left(\frac{n}{\epsilon} \log \frac{n}{\epsilon}\right)$$



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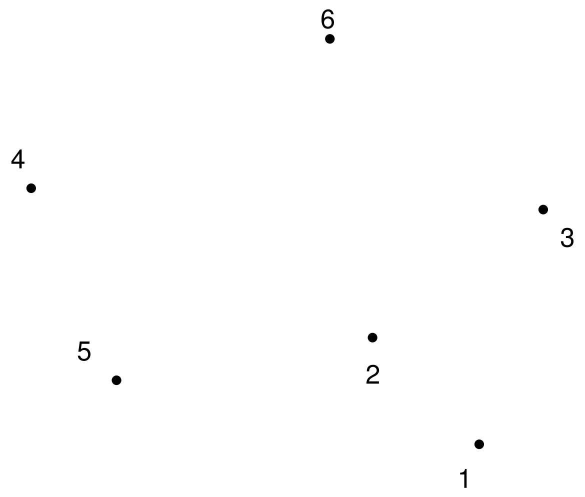

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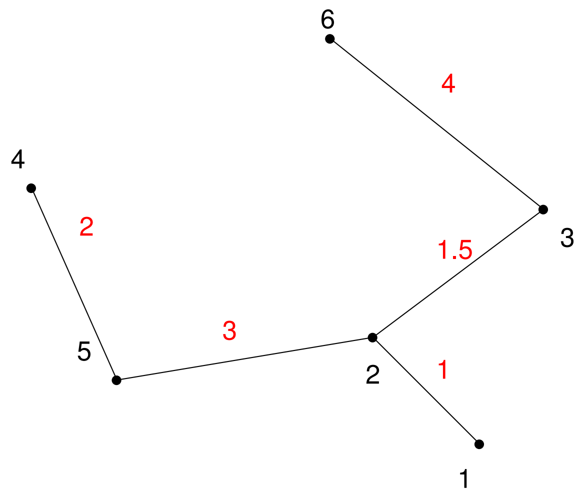

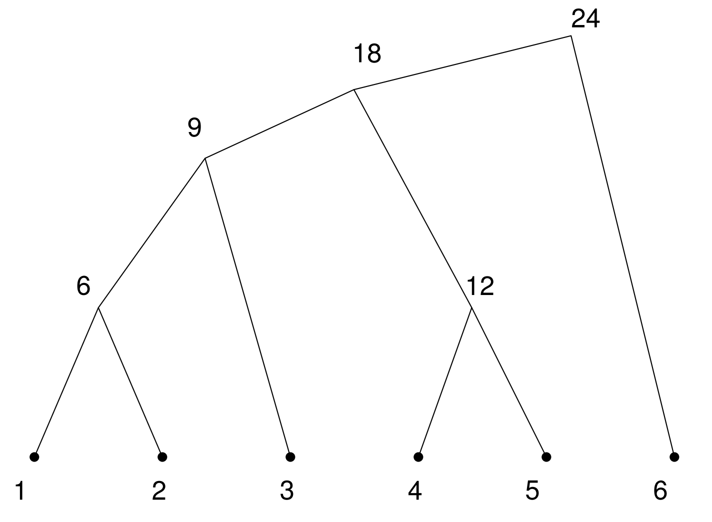

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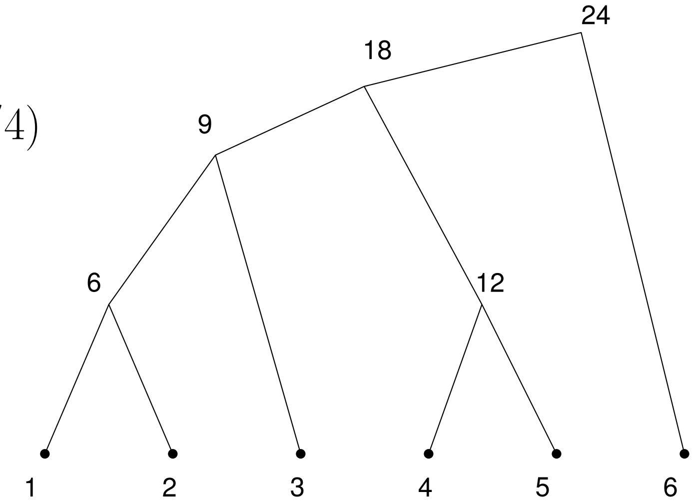

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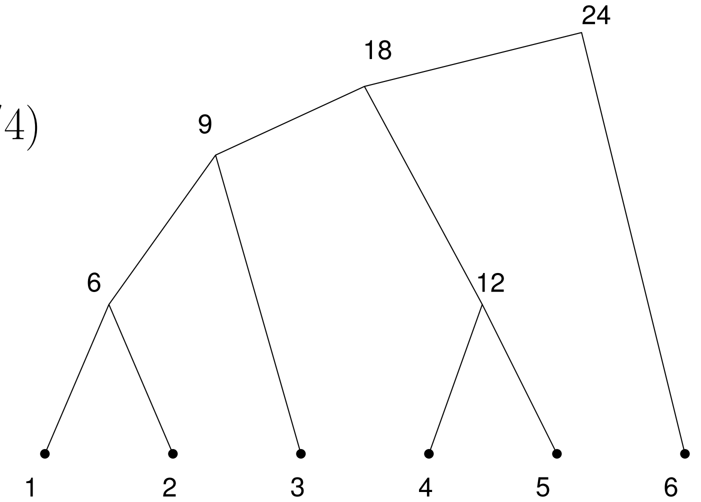

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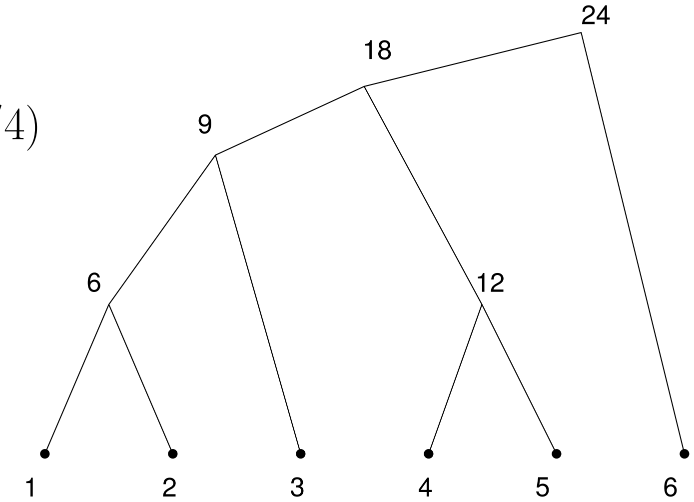

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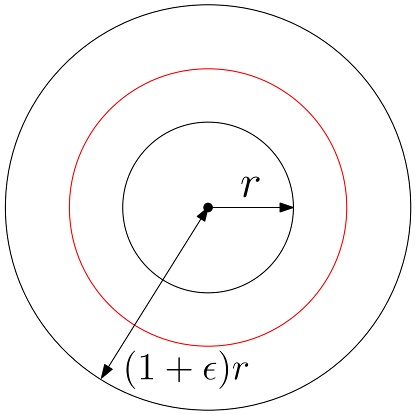

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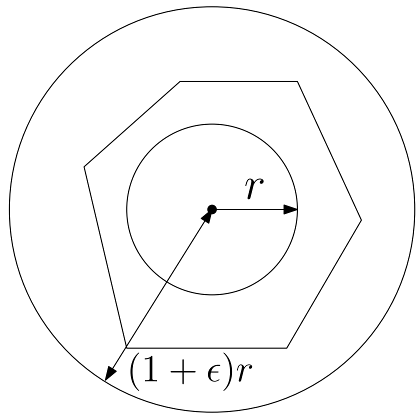

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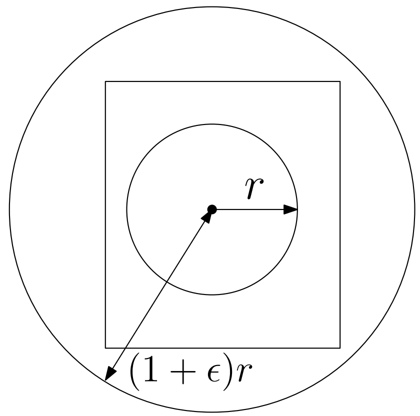

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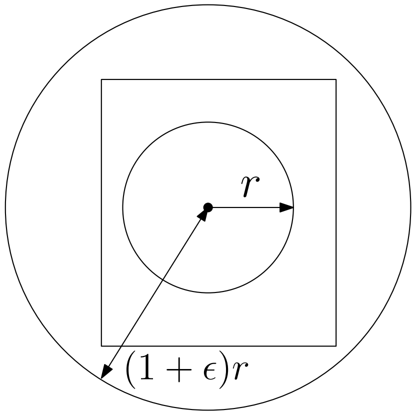

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Consider *Interval Near Neighbor* structure on approximate balls:

If $I_\approx(P, r, R, \epsilon/16)$ is a $(1+\epsilon/16)$ approximation to $I(P, r, R, \epsilon/16)$

If for point q, $I_\approx(P, r, R, \epsilon/16)$ returns a ball $(p, \alpha), \alpha \in [r, R] \implies p$ is $(1+\epsilon/4)$-ANN to q:

$$r(1+\epsilon/16)^i \leq d_P(q) \leq d(p,q) \leq r(1+\epsilon/16)^{i+1}(1+\epsilon/16) \leq (1+\epsilon/4)r$$

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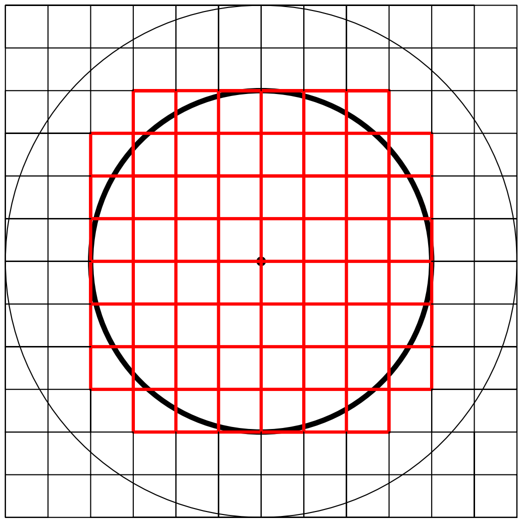
The distance between 2 points in a d-dimensional cell of size α is at most $\sqrt{\sum_{i=1}^{d} \alpha^2} = \sqrt{d}\alpha$

For a given ball, **b**(p, r), construct a grid centered at p, with cell-size 2^i, s.t. $\sqrt{d}2^i \leq \frac{(\epsilon r)}{16}$
Call, **b**$_{\approx}$ the set of cells that intersect **b**(p, r)



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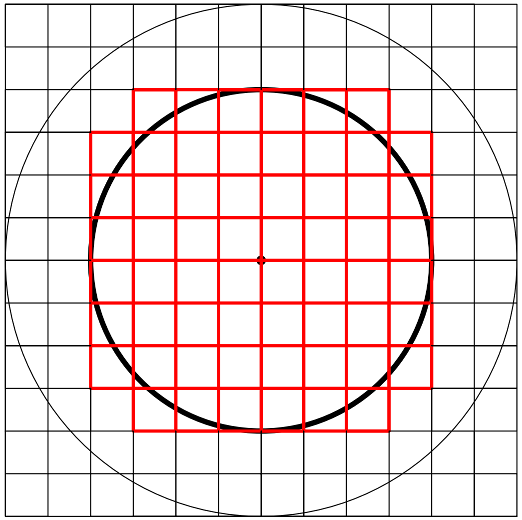

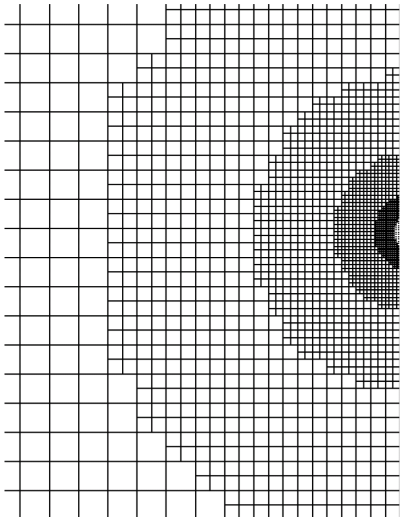

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Recall that we had a data structure with $O(\frac{n}{\epsilon} \log \frac{n}{\epsilon})$ balls. Each ball is approximated by $O(\frac{1}{\epsilon^d})$ cells
$\Rightarrow$ The overall complexity of the quad-tree is $O(N)$, where $N = O(\frac{n}{\epsilon^{d+1}} \log \frac{n}{\epsilon})$.
By noticing that there are many balls of similar sizes, we reduce the complexity to:

- Construction: $O(n\epsilon^{-d} \log^2(n/\epsilon)$ time
- Storage: $O(n\epsilon^{-d} \log(n/\epsilon)$ space
- Point location query: $O(\log(n/\epsilon))$