Homework 1: Discrete and Smooth Curves

Differential Geometry for Computer Science (Spring 2013), Stanford University

Due Monday, April 15, in class

This is the first homework assignment for CS 468. We will have assignments approximately every two weeks. Check the course website for assignment materials and the late policy. Although you may discuss problems with your peers in CS 468, your homework is expected to be your own work.

Problem 1 (20 points). *Consider the parametrized curve* $\gamma(s) := \big(a\cos(s/L), a\sin(s/L), bs/L\big)$ *for* $s \in \mathbb{R}$*, where the numbers* $a, b, L \in \mathbb{R}$ *satisfy* $a^2 + b^2 = L^2$.

(a) Show that the parameter s *is the arc-length and find an expression for the length of* γ *for* $s \in [0, L]$.

(b) Determine the geodesic curvature vector, geodesic curvature, torsion, normal, and binormal of γ.

(c) Determine the osculating plane of γ.

(d) Show that the lines containing the normal $N(s)$ *and passing through* $\gamma(s)$ *meet the z-axis under a constant angle equal to* $\pi/2$.

(e) Show that the tangent lines to γ *make a constant angle with the z-axis.*

Problem 2 (10 points). *Let* $\gamma : I \to \mathbb{R}^3$ *be a curve parametrized by arc length. Show that the torsion of* γ *is given by*

$$\tau(s) = -\frac{\langle \dot\gamma(s) \times \ddot\gamma(s), \dddot\gamma(s)\rangle}{|k_\gamma(s)|^2}$$

where $\times$ *is the vector cross product.*

Problem 3 (20 points). *Let* $\gamma : I \to \mathbb{R}^3$ *be a curve and let* $\tilde\gamma : I \to \mathbb{R}^3$ *be the reparametrization of* γ *defined by* $\tilde\gamma := \gamma \circ \phi$ *where* $\phi : I \to I$ *is a diffeomorphism. Find a formula relating the geodesic curvature vector of* $\tilde\gamma$ *to the geodesic curvature vector of* γ *and to* ϕ*. Argue that the formula you've found shows that geodesic curvature is "invariant under reparametrizations" and depends only on the "geometry" of* γ.

Problem 4 (20 points). *In this problem we will introduce you to continuous and discrete approaches to variational calculus, one of the main tools of the differential geometry toolbox.*

(a) Suppose you are given a regular plane curve $\gamma : [0,1] \to \mathbb{R}^2$*, and take* $V : [0,1] \to \mathbb{R}^2$ *to be a vector field along* γ*. Recall that the arc length of* γ *is given by*

$$s[\gamma] = \int_0^1 \|\gamma'(t)\|\, dt.$$

We can think of $\gamma(t) + hV(t)$ *to be a displacement of* γ *along* V*. Differentiate* $s[\gamma + hV]$ *to yield* $\frac{d}{dh}s[\gamma + hV]|_{h=0}$.

(b) *You can think of each V as a "variation" of the entire curve γ at once. Explain how the derivative you took in (a) can be thought of as a directional derivative of arc length in the $V : [0,1] \to \mathbb{R}^2$ "direction."*

(c) *Assume $V(0) = V(1) = 0$ and use integration by parts to write $\frac{d}{dh}s[\gamma + hV]|_{h=0} = \int_0^1 \langle V, W \rangle \, dt$ where $W : [0,1] \to \mathbb{R}^2$ can be written in terms of γ and its derivatives. Conclude by describing the variational gradient of arc length and the best direction to flow γ to increase/decrease its arc length.*

(d) *Now, assume we have a discrete curve given by a series of points $x_0, x_1, ..., x_n \in \mathbb{R}^2$. You can think of "stacking" all these points in a single vector to describe your curve as one point in $\mathbb{R}^{2n}$. Describe the arc length functional $\bar{s} : \mathbb{R}^{2n} \to \mathbb{R}$ and take its gradient to find a discrete answer paralleling your answer to (c). In particular, write an expression for the derivative of arc length with respect to x_i for $0 < i < n$, and show that its norm is $2\sin\frac{\theta}{2}$ for turning angle θ between the segments adjacent to x_i.*

Problem 5 (15 points). *Now, you will make use of your answer to the previous problem to implement a simple flow.*

(a) *Take a look at `shrinkCurve.m`. The code generates an $n \times 2$ array representing n points on a discrete two-dimensional curve. In part (d) of the previous problem, you computed the gradient of the discrete arc length $\bar{s}$. For sufficiently small Δt, one simple way to decrease the length of the curve would be to replace each point x_i with a new point $x_i' \equiv x_i - (D_{x_i}\bar{s})\Delta t$, where $D_{x_i}\bar{s}$ is the derivative of $\bar{s}$ with respect to x_i (make sure you understand why!). Implement this forward integration scheme, and make sure that if you run your code long enough the curve becomes a straight line.*

(b) *Uncomment the segment of code labeled `evil curve`. Describe the shape of the curve we have provided. Assume `nSamples` is large enough that boundary conditions affect the solution minimally for the time scale we're interested in. Describe the dependence of the behavior of our time stepping scheme from part (a) on the time step Δt.*

(c) *Suppose you are are able to choose Δt "properly," so that eventually the curve approaches a straight line.*[1] *Can we say anything about the distribution of the joints of the curve* along *the straight line? Would things be different if we had moved down the gradient of $\sum_i \|x_{i+1} - x_i\|^2$ instead?*

Problem 6 (15 points). *In this problem you will implement a part of the "Discrete Elastic Rods" paper discussed in class. Take a look at `bishopFrame.m` for starter code.*

(a) *Add code to compute the $(n-2) \times 3$ array `binormal`, which contains the Darboux vector $(\kappa\mathbf{b})_i$ for each vertex i except the first and last.*

(b) *In the second half of the script, we provide simple code for animating different initial choices of the Bishop frame $(\mathbf{u}, \mathbf{v}, \mathbf{t})$ on the first segment. Add code to fill in $\mathbf{u}$ and $\mathbf{v}$ along the rest of the curve.*

[1] Such a choice of Δt may not always be possible, and you may have to adapt Δt to the shape of the curve! We'll leave such details to CS 205.