CS 468

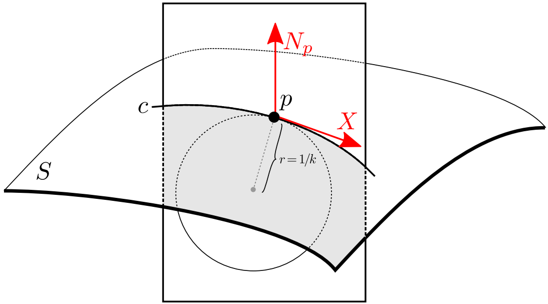
Differential Geometry for Computer Science

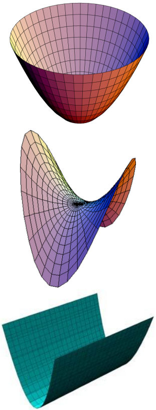
Lecture 11 — Covariant Differentiation

High-Level Summary

The extrinsic geometry of a surface.

- Variation of the unit normal vector field.
- Second fundamental form (mean and Gauss curvatures, etc.)



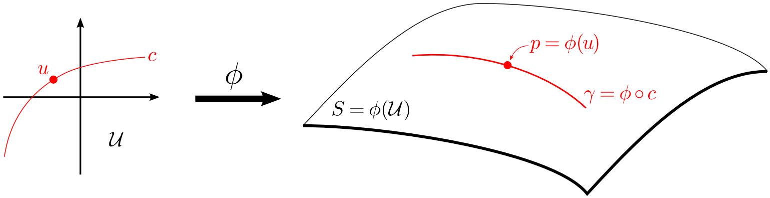


High-Level Summary

The intrinsic geometry of a surface.

- The induced metric — the Euclidean inner product **restricted** to each tangent plane.
- Pulls back under a parametrization to $g_u := [D\phi_u]^\top [D\phi_u]$.

So far we've seen that **intrinsic lengths** can be expressed via g.



$$\textit{length} := \int_0^1 \sqrt{[\dot{c}(t)]^\top \cdot g_{c(t)} \cdot [\dot{c}(t)]}\, dt \qquad \textit{length} := \int_0^1 \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt$$

Outlook

There's a loose end in the intrinsic geometry story so far:

- The equation satisfied by a length-minimizing curve $\gamma \subset S$ is called the geodesic equation:

$$\vec{k}_\gamma(t) \perp T_{\gamma(t)}S$$

- This looks completely extrinsic!

How to resolve this?

- We must show that the geodesic equation is expressible in terms of g alone. (As a system of second order ODE.)
- This involves a new topic — covariant differentiation on S.

Differentiation in Euclidean Space

- Let $V = [V^1, V^2, V^3]^\top$ be a vector in $T_p\mathbb{R}^3$ and let $c : I \to \mathbb{R}^3$ be a curve with $c(0) = p$ and $\dot{c}(0) = V$.

- Derivative of a scalar function f in the direction of a vector $V = [V^1, V^2, V^3]^\top$ is given by
$$D_V f := \left.\frac{df(c(t))}{dt}\right|_{t=0} = \sum_{i=1}^{3} V^i \frac{\partial f}{\partial x^i}$$

- Derivative of a vector field $Y(x) := [Y^1(x), Y^2(x), Y^3(x)]^\top$ in the direction of V is given by
$$D_V Y := \begin{bmatrix} D_V Y^1 \\ D_V Y^2 \\ D_V Y^3 \end{bmatrix} = \begin{bmatrix} \vdots \\ \sum_{i=1}^{3} V^i \frac{\partial Y^j}{\partial x^i} \\ \vdots \end{bmatrix}$$

Differentiation on a Surface

We can differentiate a function $f : S \to \mathbb{R}$ at a point $p \in S$ in the direction of a vector $V \in T_pS$.

- Find a curve $c : I \to S$ with $c(0) = p$ and $\frac{dc(0)}{dt} = V$.
- Then define $D_V f := \frac{d}{dt} f(c(t))\big|_{t=0}$.

Can we do the same for the derivative of a vector field $Y : S \to TS$?

- No! The vector field $\frac{d}{dt} Y(c(t))\big|_{t=0}$ is not tangent to S.

Are there alternatives?

- Is there a geometric definition on S?
- Can we use a parametrization $\phi : \mathcal{U} \to S$?
- We'd need to differentiate the coordinate vectors $E_i := D\phi(\frac{\partial}{\partial u^i})$. What about parameter independence?

Covariant Differentiation

Start with a geometric definition on S.

Let Y be a vector field on S and $V_p \in T_pS$ a vector.

$$\nabla_V Y := \left[D_V Y\right]^{\parallel}$$

Here $D_V Y$ is the Euclidean derivative $\frac{d}{dt} Y(c(t))\big|_{t=0}$ where c is a curve in S such that $c(0) = p$ and $\dot{c}(0) = V_p$.

Note: We have a relationship with the second fundamental form:

$$D_V Y = \left[D_V Y\right]^{\perp} + \left[D_V Y\right]^{\parallel} = A(V, Y)N + \nabla_V Y$$

Properties of the Covariant Derivative

As defined, $\nabla_V Y$ depends only on V_p and Y to first order along c.

Also, we have the Five Properties:

1. C^∞-linearity in the V-slot:
   $\nabla_{V_1+fV_2} Y = \nabla_{V_1} Y + f\,\nabla_{V_2} Y$ where $f : S \to \mathbb{R}$
2. $\mathbb{R}$-linearity in the Y-slot:
   $\nabla_V(Y_1 + aY_2) = \nabla_V Y_1 + a\,\nabla_V Y_2$ where $a \in \mathbb{R}$
3. Product rule in the Y-slot:
   $\nabla_V(f\,Y) = f \cdot \nabla_V Y + (\nabla_V f) \cdot Y$ where $f : S \to \mathbb{R}$
4. The metric compatibility property:
   $\nabla_V\langle Y, Z\rangle = \langle \nabla_V Y, Z\rangle + \langle Y, \nabla_V Z\rangle$
5. The "torsion-free" property:
   $\nabla_{V_1} V_2 - \nabla_{V_2} V_1 = [V_1, V_2]$

The Lie bracket

$$[V_1, V_2](f) := D_{V_1} D_{V_2}(f) - D_{V_2} D_{V_1}(f)$$

Defines a vector field, which is tangent to S if V_1, V_2 are!

The View From the Parameter Domain

Let $\phi : \mathcal{U} \to S$ be a parametrization with $\phi(0) = p$. A basis for the tangent planes $T_{\phi(u)}S$ near p is given by $E_i(u) := \frac{\partial \phi}{\partial u^i}$.

A calculation:

- Let $V_p = \sum_i a^i E_i(0)$ and $Y_{\phi(u)} := \sum_i b^i(u) E_i(u)$
- The covariant derivative computed using the Five Properties:

$$\begin{aligned}
\nabla_V Y &= \nabla_{\sum_i a^i E_i}\left(\sum_j b^j E_j\right) \\
&= \sum_{ij} a^i \nabla_{E_i}\left(b^j(u) E_j(u)\right) \\
&= \sum_{ij} a^i \left(\nabla_{E_i}(b^j) E_j) + a_i b_j \nabla_{E_i} E_j\right) \\
&= \sum_k \left(\sum_i a^i \frac{\partial b^k}{\partial u^i} + \sum_{ij} a^i b^j \Gamma^k_{ij}\right) E_k
\end{aligned}$$

The Christoffel symbols

$$\nabla_{E_i} E_j := \sum_k \Gamma^k_{ij} E_k$$

The Fundamental Lemma of Riemannian Geometry

The induced metric g and the Five Properties determines a unique covariant derivative called the Levi-Civita connection.

This relationship between g and ∇ is determined by the formula

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \quad \text{where } \Gamma_{ijk} := g(\nabla_{E_i} E_j, E_k)$$

Note: $\Gamma_{ij}^k = \sum_\ell g^{k\ell}\Gamma_{ij\ell}$ where $g^{k\ell}$ are the components of g^{-1}.

The Geodesic Equation

Recall: The geodesic equation (so far) is the extrinsic equation

$$\vec{k}_\gamma(t) \perp T_{\gamma(t)}S$$

But: We can re-express this as a purely intrinsic equation

$$\begin{aligned} \vec{k}_\gamma(t) \perp T_{\gamma(t)}S \quad &\Leftrightarrow \quad \ddot{\gamma}(t) \perp T_{\gamma(t)}S \\ &\Leftrightarrow \quad [\ddot{\gamma}(t)]^{\parallel} = 0 \\ &\Leftrightarrow \quad [D_{\dot{\gamma}}\dot{\gamma}(t)]^{\parallel} = 0 \\ &\Leftrightarrow \quad \nabla_{\dot{\gamma}}\dot{\gamma}(t) = 0 \end{aligned}$$

In the parameter domain, this is a system of second order ODEs with coefficients determined from g.

$$\gamma \text{ is a geodesic} \quad \Leftrightarrow \quad \frac{d^2\gamma^k}{dt^2} + \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^k_{ij} = 0$$

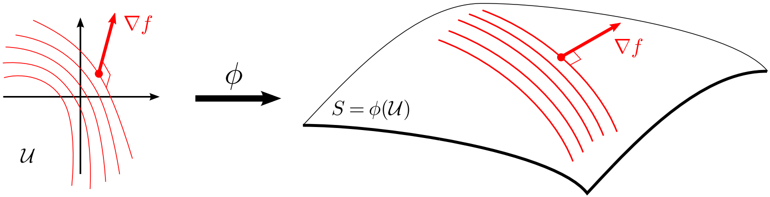
The Gradient of a Function

How does one define the gradient of a function?

- We can give a geometric definition using directional derivatives.
- Let $c : I \to S$ be a curve with $c(0) = p$ and $\dot{c}(0) = V$. Then

$$\left.\frac{df(c(t))}{dt}\right|_{t=0} := \langle \nabla f(p), V \rangle$$

- In Euclidean space, $\nabla f(p) = [Df_p]^\top$. But in the parameter domain, $\langle \cdot, \cdot \rangle \to g$ so $\nabla f = g^{-1} \cdot Df$.



Vector Analysis Operators

The important covariant differential operators on a surface:

	Geometric Definition	In the parameter domain
The gradient of $f : S \to \mathbb{R}$	∇f s.t. $D_V(f) := \langle \nabla f, V \rangle$	$[\nabla f]^i := \sum_j g^{ij} \frac{\partial f}{\partial u^j}$
The divergence of the v.fld. X	$\nabla \cdot X := \sum_j \langle \nabla_{E_j} X, E_j \rangle$ where E_i is an ONB	$\nabla \cdot X := \sum_i \left[\frac{\partial X^i}{\partial u^i} + \sum_j \Gamma^i_{ij} X^j \right]$
The Laplacian of $f : S \to \mathbb{R}$	$\Delta f := \nabla \cdot (\nabla f)$	$\Delta f := \sum_{ij} g^{ij} \left[\frac{\partial^2 f}{\partial u^i \partial u^j} + \Gamma^k_{ij} \frac{\partial f}{\partial u^k} \right]$

Note: We have an integration by parts formula:

$$\int_S f \, \nabla \cdot X \, dA = - \int_S \langle \nabla f, X \rangle \, dA + \int_{\partial S} f \langle X, \vec{n}_{\partial S} \rangle \, d\ell$$