CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 11 Supplement

1. The Flow of a Vector Field

Let V be a smooth vector field on $\mathbb{R}^3$, so that $V(x) \in T_x\mathbb{R}^3$ for every $x \in \mathbb{R}^3$. In other words, each $x \in \mathbb{R}^3$ has a vector $V(x)$ attached to it. A *streamline* of V is a curve $\gamma : I \to \mathbb{R}^3$ whose tangent vector coincides with Y everywhere. More precisely, the streamline through a given x satisfies the equation

$$\frac{d\gamma}{dt} = V(\gamma(t)) \qquad \text{with} \qquad \gamma(0) = x\,. \tag{1}$$

If we write $\gamma(t) := (x^1(t), x^2(t), x^3(t))$ as well as $V(x) := [V^1(x), V^2(x), V^3(x)]^\top$ and $x := (x^1, x^2, x^3)$ then we can see that the above equation is in fact a system of three first-order ODEs for $x^1(t)$, $x^2(t)$, $x^3(t)$. That is,

$$\frac{dx^i}{dt} = V^i(x) \;\text{ for } i = 1,2,3 \qquad \text{with} \qquad x(0) = x$$

By the Existence and Uniqueness Theorem for ODEs, there always exists a unique solution of these equations on a uniform time interval I under mild regularity conditions on V.

We can now define the *flow* of the vector field V. This is the one-parameter family of diffeomorphism $\Phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ where $t \in I$ defined as follows. Given $x \in \mathbb{R}^3$, then $\Phi_t(x)$ is equal to the point in $\mathbb{R}^3$ obtained by moving forward t units of time along the streamline through x. Symbolically,

$$\Phi_t(x) := \gamma(t) \qquad \text{where } \gamma \text{ solves (??)}.$$

Remark: The above concepts extend in the following way to surfaces. The definition of steamline is unchanged. The equation satisfied by a streamline still looks like (??) but it must its solvability must be analyzed in a parameter domain, where it becomes a system of two first-order ODE in the parameters. You can prove that the solution varies in a compatible manner when the parametrization is changed. The definition of flow is unchanged.

2. Lie Differentiation

Given a vector field Y on $\mathbb{R}^3$ along with another vector field V and its flow Φ_t, we can compute the *Lie derivative* of Y in the direction of V. We find this by differentiating Y along the flow generated by V. Namely, to compute the Lie derivative at x, we take the vector field Y at a point further ahead from x along the flow of V and pull it back under the differential of the inverse flow to x. Then we subtract the pulled-back vector from $Y(x)$, divide by t and take the limit. Symbolically,

$$\mathcal{L}_V Y(x) := \lim_{t\to 0} \frac{[D\Phi_{-t}]_x\big(Y \circ \Phi_t(x)\big) - Y(x)}{t} = \frac{d}{dt}\left([D\Phi_{-t}]_x Y \circ \Phi_t(x)\right)\Big|_{t=0}.$$

Remark: This idea extends more or less directly to surfaces.

3. The Lie Bracket

Let V and W be two vector fields on $\mathbb{R}^3$. We define their *Lie bracket* as the commutator of the directional derivative operators associated to these vector fields. In other words, given any test function $\xi : \mathbb{R}^3 \to \mathbb{R}$, we define

$$[V, W]\xi := D_W\big(D_W(\xi)\big) - D_W\big(D_V(\xi)\big)$$

where as usual, $D_V(\xi)$ evaluated at a point p means "take any curve $c : I \to \mathbb{R}^3$ with $c(0) = p$ and $\dot{c}(0) = V(p)$, and then compute $\frac{d}{dt}\xi(c(t))\big|_{t=0}$." So to compute $[V,W]\xi$, you just perform this operation twice, and twice more in the opposite order, and subtract the results.

Proposition 1 (Important Properties of the Lie Bracket)**.** *Let V and V_2 be vector fields on $\mathbb{R}^3$.*

1. *$[V,W]$ is a vector field on $\mathbb{R}^3$. If $V := [V^1, V^2, V^3]^\top$ and $W := [W^1, W^2, W^3]^\top$ then $[V,W]$ has components given by*

$$[V,W]^j = \sum_{i=1}^{3} \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \qquad j = 1,2,3$$

2. *The Lie bracket and the Lie derivative are related via $\mathcal{L}_V W = [V,W]$.*

3. *Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth mapping. Then the differential of ϕ preserves Lie brackets. In other words,*

$$D\phi_x\big([V,W]\big) = [D\phi_x(V), D\phi_x(W)]$$

4. *Suppose V has flow Φ_t and W has flow Ψ_t. Then $[V,W] = 0$ if and only if these flows commute, i.e. $\Phi_s \circ \Psi_t = \Psi_t \circ \Phi_s$ for all s,t.*

Discussion. The way we have defined it, the Lie bracket is a differential operator on functions. But it is not necessarily a vector field because vector fields are a very specific type of differential operator. A vector field, when acting as a differential operator via directional differentiation, depend linearly on the derivative matrix of the function it acts upon. We have to verify that this is the case for the Lie bracket — which we do by calculating the formula of (1). This formula, which is straightforward to obtain, show the right kind of dependence on the derivatives of ξ. The formula of (2) uses the definition of $\mathcal{L}_V W$ as well as the equations satisfied by the flow of V. The formula of (3) can be proved by substituting the formula for $D\phi_x(W)$ into the formula of (1). At a crucial point, the derivation of the formula of (3) requires the identity $\frac{\partial^2 \phi^k}{\partial x^i \partial x^j} = \frac{\partial^2 \phi^k}{\partial x^j \partial x^i}$. Finally, we can interpret the the flow commutator in (4) as follows. First, "move t units along the streamlines of V then s units along the streamlines of W." Then start again and "move s units along the streamlines of W then t units along the streamlines of V." You obtain the same final result if and only if $[V,W] = 0$. □

4. Lie Brackets on Surfaces

The results from the previous subsection carry over to the surface case more or less exactly. The formula of part (1) of the proposition is valid in every parameter plane. Independence of parametrization (covariance, actually, since we're dealing with vector fields) follows from part (3). In fact, part (3) has two important consequences.

Corollary 2. *Let S be a surface.*

1. *Let V,W be two tangent vector fields to S. Then $[V,W]$ is a tangent vector field to S as well.*

2. *Let $\phi : \mathcal{U} \to \mathbb{R}^3$ be a parametrization and let $E_i := \frac{\partial \phi}{\partial u^i}$ be the coordinate vector fields. Then $[E_i, E_j] = 0$.*

Sketch of Proof. Both results follow from part (3). Our first assertion holds because we *defined* the tangent space T_pS as the image of $D\phi_p$. Our second assertion holds because $E_i = D\phi_p([0,\ldots,1,\ldots,0]^\top)$ and clearly the Lie bracket of any pair of constant vectors vanishes. □

5. The Torsion-Free Property of the Levi-Civita Connection

The fifth property satisfied by the Levi-Civita connection of a surface S was incorrectly stated in class. With typos corrected, the property reads:

$$\nabla_V W - \nabla_W V = [V, W] \qquad \text{for all vector fields } V, W \text{ on } S.$$

It is not a simple matter to get intuition for this formula. It certainly holds when we are in Euclidean space and ∇ is replaced by D. So the fifth property above is a natural extension to surfaces. Morally speaking, it is related to the idea the "second partial derivatives of functions commute." So if $\xi : S \to \mathbb{R}$ is a function and E_i, E_j are coordinate vector fields, we have

$$\big(\nabla_{E_i} E_j - \nabla_{E_j} E_i\big)(\xi) = 0\,.$$