CS 468 LECTURE 15: ISOMETRIES, RIGIDITY, AND CURVATURE

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1. Outline

In this lecture we will introduce the Riemann curvature tensor, an intrinsic object which is, in general, quite unwieldy. However, in two dimensions (i.e., on surfaces), the Riemann curvature tensor is essentially one number, the Gauss curvature. Thus we will see that the Gauss curvature can actually be characterized intrinsically.

To get to the Riemann curvature tensor, we will first return to our discussion of the exponential map from Lecture 9. The exponential map is an intrinsic object, and we will use it to parameterize small neighborhoods of a surface. We proved in a previous lecture that the exponential map is a diffeomorphism near the origin, and in this lecture we will use this diffeomorphism to construct a parametrization of a surface and talk about the induced metric in the corresponding coordinates.

After introducing the Riemann curvature tensor, we will then cover the Theorema Egregium, which describes (in two dimensions) Gauss curvature as the important local intrinsic invariant of a surface.

We will close the lecture with a discussion of isometries and isometric invariance.

2. The Exponential Map

We first recall the definition of the geodesic exponential map of a surface S at a point $p \in S$. Define $\exp_p : \mathcal{U} \to S$ by $\exp_p(V) := \gamma(1)$ for $V \in \mathcal{U} \subset T_pS$, where γ is the unique geodesic through p in direction V, i.e., with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$. Notice that the exponential map corresponds to traveling a unit distance along this geodesic.

The proof of the existence and uniqueness of such a geodesic was outlined in a previous lecture. The geodesic equations were a system of second-order ODEs, and a theorem from the theory of ODEs guarantees the existence and uniqueness of a solution to this system given the two initial conditions $\gamma(0)$ and $\dot{\gamma}(0)$. This theorem, however, only guarantees a local solution to the geodesic equations. We need norm of the argument V to be sufficiently small in order to guarantee the existence and uniqueness of the desired geodesic corresponding to V, i.e., to guarantee that the exponential map is well-defined. So we choose an open domain $\mathcal{U} \subset T_pS$ small

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enough such that the exponential map is well-defined.

Recall from a previous lecture that the differential of the exponential map at the origin is the identity map, which is of course invertible, so the exponential map itself is locally invertible by the inverse function theorem. So if we take the domain $\mathcal{U}$ to be a small enough ball, the exponential map $\exp_p : \mathcal{U} \to \mathcal{V} \subset S$ is a diffeomorphism, where $\mathcal{V} = \exp_p(\mathcal{U})$.

The following are some important facts relating to the exponential map:

- Although we will not prove this, we can assume without loss of generality that we can take the domain $\mathcal{U}$ such that $\mathcal{U}$ and its image $\mathcal{V}$ are geodesically convex, i.e., the geodesic between any two points in $\mathcal{U}$ (or $\mathcal{V}$) lies within $\mathcal{U}$ (or $\mathcal{V}$).
- The curve $t \mapsto \exp_p(tV)$ is a geodesic for each $V \in \mathcal{U}$. This follows directly from the definition of the exponential map.

3. Geodesic Normal Coordinates

Idea: given our diffeomorphism from the tangent space to the surface itself, we can construct coordinates (i.e., a local parameterization) for the surface. We have a mapping from the tangent plane into the surface, and we want a mapping from $\mathbb{R}^2$ into the surface (i.e., a parameterization).

Choosing a basis for the tangent space gives us an obvious isomorphism between the tangent space and $\mathbb{R}^2$.

Let e_1, e_2 form a basis for T_pS. Without loss of generality (by applying the Gram-Schmidt process), we can assume that this is an orthonormal basis. Let $\psi : \mathbb{R}^2 \to T_pS$, defined by $(x^1, x^2) \mapsto x^1 e_1 + x^2 e_2$, be the obvious vector space isomorphism between $\mathbb{R}^2$ and the tangent space. Then this gives a local paramaterization of the surface $\phi = \exp_p \circ \psi$, where $\phi(x^1, x^2) = \exp_p(x^1 e_1 + x^2 e_2)$ for a vector $(x^1, x^2) \in \mathbb{R}^2$.

We restrict the domain of ϕ to only a ball of radius r about the origin, $B_r(0)$, where r is small enough such that $\psi(B_r(0)) \subset \mathcal{U}$.

We list some important properties of this construction:

(1) Straight lines through the origin in $B_r(0)$ are geodesics.
(2) The induced metric is Euclidean (through first order) at the origin in the parameter domain $B_r(0)$. More precisely, $g_{ij}(x) = \delta_{ij} + \mathcal{O}(\|x\|^2)$ for $x \in B_r(0)$, where δ_{ij} is the Kronecker delta. This expression is an abbreviated Taylor series expansion of the metric in geodesic normal coordinates.
(3) The Christoffel symbols vanish at the origin in the parameter domain $B_r(0)$.

To see (1), note that straight lines through the origin in the parameter domain $B_r(0)$ map under ψ to straight lines in the tangent space. Of course, letting l be the straight line through the origin given by $l(t) = t(x_0^1, x_0^2)$, we have $\psi \circ \gamma(t) = t(x_0^1 e_1 + x_0^2 e_2)$, so $\phi \circ l$ is given by $t \mapsto \exp_p(tV)$ where $V = x_0^1 e_1 + x_0^2 e_2 \in \mathcal{U}$.

Hence by our previous observation, straight lines through the origin map under ϕ to geodesics. We cannot, however, make any such statement about straight lines that do not pass through the origin.

We will prove properties (2) and (3) together. First recall from above that

$$[D\exp_p]_{(0,0)} = \mathrm{id}$$

and from our definition of ϕ that

$$\phi(x^1, x^2) = \exp_p(x^1 e_1 + x^2 e_2).$$

So it follows that

$$\frac{\partial \phi}{\partial x^1}(0,0) = [D\exp_p]_{(0,0)}(e_1) = e_1,$$

$$\frac{\partial \phi}{\partial x^2}(0,0) = [D\exp_p]_{(0,0)}(e_2) = e_2.$$

Hence from the definition of the induced metric,

$$g_{ij}(0,0) = \left\langle \frac{\partial \phi}{\partial x^1}(0,0), \frac{\partial \phi}{\partial x^2}(0,0) \right\rangle = \delta_{ij}.$$

So up through zeroth order, the induced metric is Euclidean. We can see from the proof that we have essentially forced this to be true by choosing e_1, e_2 to be orthonormal. The more interesting content of the second property above is that the induced metric is actually Euclidean through first order.

For this, it remains to show that $\frac{\partial g_{ij}}{\partial x^k}(0,0) = 0$ for all i, j, k. We will actually see that it will suffice to show that the Christoffel symbols $\Gamma^k_{ij}(0,0) = 0$ for all i, j, k. This fact is the third property listed above, so we will prove both simultaneously. To begin, recall the definition of these Christoffel symbols: $\nabla_{E_i} E_j = \sum_k \Gamma^k_{ij} E_k$, where the E_i are the coordinate vector fields $\frac{\partial \phi}{\partial x^i}$.

So, using the metric compatibility property of the covariant derivative, we see that

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial}{\partial x^k} \langle E_i, E_j \rangle = \langle \nabla_{E_k} E_i, E_j \rangle + \langle E_i, \nabla_{E_k} E_j \rangle = \Gamma^j_{ki} + \Gamma^i_{kj}.$$

The final equality above uses the definition of the Christoffel symbols and the orthonormality of the E_i.

So indeed it will suffice to show that $\Gamma^k_{ij}(0,0) = 0$ for all i, j, k.

Recall that straight lines through the origin in the parameter domain map to geodesics on the surface. So letting l be the straight line through the origin given by $l(t) = tV$ for some vector V, we have that l must satsify the geodesic equation, i.e., $\nabla_{\dot{l}} \dot{l}(t) = 0$. Since $\dot{l} \equiv V$, we see that $\nabla_V V = 0$ at the origin for all vectors V

in the parameter domain (because all straight lines through the origin contain the origin). So let $V = E_i + E_j$. Then $\nabla_{E_i+E_j}(E_i + E_j) = 0$ at the origin.

So $\nabla_{E_i}E_i + \nabla_{E_i}E_j + \nabla_{E_j}E_i + \nabla_{E_j}E_j = 0$ at the origin by linearity in both slots of the covariant derivative.

Since the first and last terms in the left hand side above are zero at the origin, it follows that $\nabla_{E_i}E_j + \nabla_{E_j}E_i = 0$ at the origin.

The torsion-free property of the covariant derivative implies that $\nabla_{E_i}E_j = \nabla_{E_j}E_i + [E_i, E_j]$. Note that the Lie bracket term $[E_i, E_j] = 0$ because the E_k are the coordinate vector fields (the interested reader can see the lecture 11 supplement for more details on the Lie bracket and the proof of this fact). So $\nabla_{E_i}E_j = \nabla_{E_j}E_i$.

Now we have that $2\nabla_{E_i}E_j = 0$ at the origin. Hence $\nabla_{E_i}E_j = \sum_k \Gamma_{ij}^k E_k = 0$, so all the coordinates Γ_{ij}^k of this vector are zero at the origin, as was to be shown. This completes the proof of the above properties.

4. Local Rigidity

The following discussion of rigidity is local in that it applies to a neighborhood of a point on a surface. We have found coordinates that make the induced metric Euclidean (through first order) near a point. An early question in differental geometry which motivated Riemann was whether we could do better than this. Can we change coordinates to make the metric trivial, i.e., Euclidean everywhere?

The answer was thought to be "no" for a long time. This is the so-called "mapmaker's problem." The existence of a parameterization that gives a trivial metric would mean that measuring lengths on the surface is equivalent to measuring them in the parameter domain, i.e., we could have a map of, say, the Earth which does not distort lengths. It was conjectured for some time that a map that does not distort lengths is an impossibility. If this is indeed impossible, we would like to know mathematically why this is the case, and whether we can at least we find coordinates that are Euclidean to second order near a point.

The answer to these questions is that we indeed cannot (in general) find a coordinates the are Euclidean even to second order. We will only give a very rough sketch of the proof of this fact. (The interested reader should see Spivak's *A Comprehensive Introduction to Differential Geometry* for a complete proof of this fact.) The basic idea is that we can come up with a system of equations in which the parameterization is unknown. This system essentially writes the components of the metric in terms of the unknown parameterization and set them equal to δ_{ij}. It turns out that this system is overdetermined. There are two unknown functions (the parameterizations), and there are "lots" of equations (three of them in dimension two).

So we check the "integrability conditions" of this system, i.e., the conditions that must hold for a solution to exist. These conditions turn out to be expressible in terms of the Christoffel symbols (and their derivatives):

$$0 = \frac{\partial \Gamma^s_{jk}}{\partial x^i} - \frac{\partial \Gamma^s_{ik}}{\partial x^j} + \Gamma^t_{jk}\Gamma^s_{it} - \Gamma^t_{ik}\Gamma^s_{jt}$$

for all i, j, k, s.

The Christoffel symbols are combinations of first derivatives of the metric, so we see that some combination of second derivatives of the metric must vanish if the mapmaker's problem is to have a solution. The right hand sides of these equations can be understood as quantifying the local rigidity of a surface. In general, curved surfaces are "rigid" in that they cannot be flattened out without distorting distances. If the above expressions are zero, then the surface is "flexible" enough to be flattened (into a map, for example) without distortion of lengths.

5. Gauss' Totally Awesome Theorem

We define the Riemann curvature $(3, 1)$-tensor of S by

$$\mathrm{Rm}(X, Y, Z) := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields. Notice the covariant derivative of a vector field is a vector field, so Rm, a $(3, 1)$ tensor, indeed returns a vector field.

We can expand on a basis $\{e_i\}$, giving:

$$\mathrm{Rm} = \sum_{ijks} R^s_{ijk} \omega^i \otimes \omega^j \otimes \omega^k \otimes E_s,$$

where R^s_{ijk} is the s-component of $\mathrm{Rm}(e_i, e_j, e_k)$. Computations that are omitted here (deferred to the homework) show that

$$R^s_{ijk} = \frac{\partial \Gamma^s_{jk}}{\partial x^i} - \frac{\partial \Gamma^s_{ik}}{\partial x^j} + \Gamma^t_{jk}\Gamma^s_{it} - \Gamma^t_{ik}\Gamma^s_{jt}.$$

Note that the right hand side of this equation is the same expression from the integrability conditions discussed above. This relationship can be viewed as a motivation for the definition of the Riemann curvature tensor.

We will provide another roughly intuitive motivation for the Riemann curvature tensor. In Euclidean space, "second partial derivatives of vector quantities commute," i.e., $\nabla_{e_i}\nabla_{e_j} = \nabla_{e_j}\nabla_{e_i}$, where the e_k are standard basis vectors. In non-Euclidean space, Rm is not identically zero and can be viewed as measuring the failure of commutativity of $\nabla_{e_i}\nabla_{e_j}$ and $\nabla_{e_j}\nabla_{e_i}$ (noting that the Lie bracket of coordinate vector fields is zero).

The Theorema Egregium ("Totally Awesome Theorem") of Gauss relates the very abstractly constructed Riemann curvature tensor to the more concrete second fundamental form. This is significant because the Riemann curvature tensor is an intrinsic object, while the second fundamental form is extrinsic. With the second fundamental form denoted by A, the theorem states that

$$R^s_{ijk} + (A_{jk}A^s_i - A_{ik}A^s_j) = 0$$

where $A^s_i = \sum_t g^{st}A_{it}$.

Derivation: Let $\overline{\nabla}$ be the covariant derivative in Euclidean space. Recall that if we define two vector fields X, Y and on a surface and let N be the unit normal vector field, then

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^{\parallel} + (\overline{\nabla}_X Y)^{\perp} = \nabla_X Y + A(X,Y)N$$

by the definition of the covariant derivative and the characterization of the second fundamental form given in Lecture 11.

Now for basis vector fields E_i:

$$\overline{\nabla}_{E_j}\overline{\nabla}_{E_i}E_k = \overline{\nabla}_{E_i}\overline{\nabla}_{E_j}E_k,$$

so it follows that

$$\begin{aligned} 0 &= \left\langle \overline{\nabla}_{E_j}\overline{\nabla}_{E_i}E_k - \overline{\nabla}_{E_i}\overline{\nabla}_{E_j}E_k, E_l \right\rangle \\ &= \left\langle \overline{\nabla}_{E_j}\left(\nabla_{E_i}E_k + A(E_i,E_k)N\right), E_l \right\rangle - \left\langle \overline{\nabla}_{E_i}\left(\nabla_{E_j}E_k + A(E_j,E_k)N\right), E_l \right\rangle . \end{aligned}$$

The term on the left can then be written using the product rule for the covariant derivative:

$$\left\langle \overline{\nabla}_{E_j}\left(\nabla_{E_i}E_k\right) + \left(E_j \cdot A(E_i,E_k)\right)N + A(E_i,E_k)\overline{\nabla}_{E_j}N, E_l \right\rangle$$

where the "·" indicates the action of the vector field on a function by directional derivation. We can then rewrite this expression, using the facts that $N \perp E_l$ and $A(E_j, E_l) = -\left\langle \overline{\nabla}_{E_j}N, E_l \right\rangle$, as

$$\begin{aligned} &\left\langle \overline{\nabla}_{E_j}\nabla_{E_i}E_k, E_l \right\rangle - A(E_i,E_k)A(E_j,E_l). \\ =\ &\left\langle \nabla_{E_j}\nabla_{E_i}E_k, E_l \right\rangle + \left\langle \left(\overline{\nabla}_{E_j}\left(\nabla_{E_i}E_k\right)\right)^{\perp}, E_l \right\rangle - A(E_i,E_k)A(E_j,E_l) \\ =\ &\left\langle \nabla_{E_j}\nabla_{E_i}E_k, E_l \right\rangle - A(E_i,E_k)A(E_j,E_l). \end{aligned}$$

So we have shown that

$$0 = \left\langle \nabla_{E_j}\nabla_{E_i}E_k, E_l \right\rangle - A(E_i,E_k)A(E_j,E_l) - \left\langle \overline{\nabla}_{E_i}\left(\nabla_{E_j}E_k + A(E_j,E_k)N\right), E_l \right\rangle ,$$

and by a similar computation, we see that the rightmost term in this equation can be written

$$\left\langle \nabla_{E_i}\nabla_{E_j}E_k, E_l \right\rangle - A(E_j,E_k)A(E_i,E_l),$$

so we have:

$$\left\langle \nabla_{E_j}\nabla_{E_i}E_k, E_l\right\rangle - \left\langle \nabla_{E_i}\nabla_{E_j}E_k, E_l\right\rangle = A(E_i, E_k)A(E_j, E_l) - A(E_j, E_k)A(E_i, E_l).$$

A bit more calculation, omitted here, transforms the left hand side into the curvature tensor and the right hand side into the expression involving the second fundamental form in the statement of the the theorem.

6. Interpretation

We can obtain a Riemann curvature $(4, 0)$-tensor from the $(3, 1)$-tensor defined above with coordinates $R_{ijkl} := \sum_s g_{ls} R^s_{ijk}$.

In two dimensions, there is essentially (up to sign) only one nonzero number that can be obtained among all choices of i, j, k, l. We write this number: $R_{1212} = -(A_{11}A_{22} - A_{12}^2)$. (We could also consider R_{2112}, for example.[1]) In an orthonormal basis, $R_{1212} = -\det A$, and of course $\det A$ is the Gauss curvature. Therefore, the Gauss curvature is in fact an intrinsic quantity.

7. Isometries

An isometry is a mapping from a surface to another surface (or itself) which preserves the metric at corresponding points. In other words, if S and S' are surfaces with metric g and g', then the surfaces are isometric if there exists $\phi : S \to S'$ such that for all $X_p, Y_p \in T_pS$ and all $p \in S$, we have that

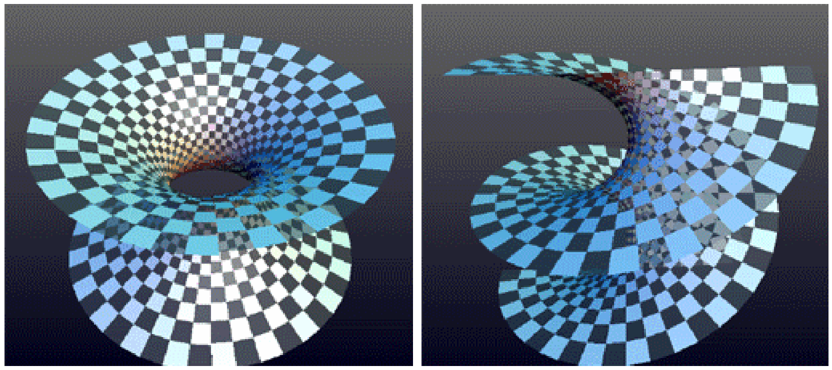
$$g'(D\phi(X_p), D\phi(Y_p)) = g(X_p, Y_p).$$

Notice that $D\phi$ pushes forward tangent vectors from T_pS to $T_{\phi(p)}S'$, so our precise definition coincides with our initial notional description of isometry. We can understand an isometry as preserving the intrinsic geometry at corresponding points.

A simple example of an isometry is the isometry induced by a rigid motion of $\mathbb{R}^3$.

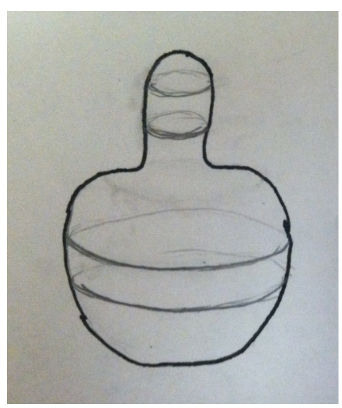
Not all isometries are induced by rigid motions. The catenoid and the helicoid (shown at left and right below, respectively) are isometric. The isometry between these two surfaces is fairly complicated, but intrinsically, the surfaces are the same. (Roughly speaking, the little “squares” in one image map to nearly identical “squares” in the other image.)

[1]Yeah, it’s a Rush reference.



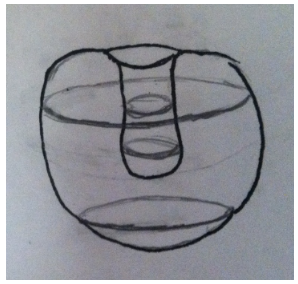
Let us consider a vertically-oriented cylinder (of infinite vertical extent). A cylinder clearly has some isometries. We can map the cylinder to itself isometrically by vertical translation. We can also rotate the cylinder. These maps are both induced by rigid motions of $\mathbb{R}^3$. We can think of the cylinder as a wrapped plane (or a wrapped piece of paper, which is not permitted to deform intinsically). We can come up with a wide variety of similar developable surfaces by gluing together the opposite ends of a "sheet of paper" which may not enjoy rotational symmetry like a cylinder. However, such surfaces have a corresponding isometry given by traveling around the surface without moving in the vertical direction. This isometry, however, is not given by a rigid motion of $\mathbb{R}^3$.

As another example, consider the following "amphora" or "bowling-pin" surface of revolution:



The symmetry of this shape allows for a family of isometries induced by rigid motions (rotations and reflections). We casually note here the difference between a continuous family of isometries like a family of rotations and a discrete isometry like a reflection.

Now consider the surface obtained by "popping" the neck of the amphora inward:



If the surface near the transition to the neck is flat enough (i.e., to second order), then the induced metric cannot "tell the difference" between these two surfaces. So we have a discrete isometry which is not induced by any rigid motion.

Isometries are very rare. Riemann curvature is invariant under isometry, so a necessary condition for a mapping to be an isometry is that the two surfaces in question have the same Riemann curvature tensor at corresponding points.