CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 15: Isometries and Rigidity

Isometries and local isometries.

- Definition of *isometric surfaces*: two surfaces M and N are isometric if there exists a mapping $\phi : M \to N$ so that $\langle D\phi(X_p), D\phi(Y_p)\rangle = \langle X_p, Y_p\rangle$ for tangent vectors $X_p, Y_p \in T_pM$ and all $p \in M$.
- Local isometries and one-parameter families of isometries and infinitesimal isometries.
- Preservation of the metric tensor. Equations satisfied by infinitesimal isometries.
- Examples of isometric surfaces and surfaces for which there are isometric deformations. Example of a local but not global isometry. Extrinsic isometries induce intrinsic ones; example of an intrinsic isometry that is not induced from an extrinsic one.

Rigidity.

- Examples of isometric surfaces and surfaces for which there are isometric deformations. Example of a local but not global isometry. Extrinsic isometries induce intrinsic ones; example of an intrinsic isometry that is not induced from an extrinsic one. Etc.
- Some rigidity theorems and open questions.

Gauss's Theorema Egregium — a local rigidity theorem.

- Derivation of the Gauss equation based on $0 = \bar{\nabla}_Y \bar{\nabla}_X - \bar{\nabla}_X \bar{\nabla}_Y$ for coordinate vector fields in $\mathbb{R}^3$. Show that the determinant of the second fundamental form arises on one side of the equation.
- The other side of the equation is $R(X, Y, Z, W) := \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, W\rangle$ for coordinate vector fields X, Y and tangential vector fields Z, W. Argue that this is *intrinsic*.
- This can be reduced to an expression involving the Christoffel symbols. Thus we have an intrinsic quantity!
- Note the symmetries of R... There's only one number there!

Gauss-Bonnet theorem — a global rigidity theorem.

- The covariant derivative $\nabla_{\dot{c}}\dot{c}$ of a curve $c : I \to M$ and the *algebraic value* of the covariant derivative or an arc-length parametrized curve — geodesic curvature.
- Angle between two vector fields along a curve; between the tangent vector and a parallel vector field; between a vector field and a coordinate vector field (doC Prop 4-4-3), between the acceleration vector and a coordinate vector field — geodesic curvature.
- Piecewise regular curves, exterior angles at singularities.
- The local Gauss-Bonnet theorem (application of the theorem of turning tangents).
- Reminder — triangulations and the Euler characteristic of a surface.
- Sketch of the proof of the global Gauss-Bonnet theorem.
- A discrete version — local Gauss-Bonnet theorem in a vertex one-ring on a triangle mesh.