CS 468

Differential Geometry for Computer Science

Lecture 15 — Isometries, Rigidity and Curvature

Outline

- Geodesic normal coordinates
- Local rigidity — Gauss curvature and the Theorema Egregium
- Isometries and isometry invariance
- Global rigidity — Gauss-Bonnet theorem

The Exponential Map

Recall: The geodesic exponential map of a surface S at $p \in S$ is the mapping $\exp_p : T_pS \to S$ defined by

$$\exp_p(V) := \gamma(1)$$

where γ is the unique geodesic through p in direction V.

Key facts:

- There are open sets $\mathcal{U} \subseteq T_pS$ containing the origin and $\mathcal{V} \subseteq S$ containing p so that $\exp_p : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism.
- W.l.o.g. $\mathcal{U}$ and $\mathcal{V}$ are geodesically convex.
- The curve $t \to \exp_p(tV)$ is a geodesic for each $V \in \mathcal{U}$.

Geodesic Normal Coordinates

We can use $\exp_p$ to create local coordinates near $p \in S$.

- Choose an orthonormal basis e_1, e_2 for T_pS.
- Choose r so that $x^1 e_1 + x^2 e_2 \in \mathcal{U}$ for all $(x^1, x^2) \in B_r(0) \subseteq \mathbb{R}^2$.
- Define $\phi : B_r(0) \to S$ by $\phi(x^1, x^2) := \exp_p(x^1 e_1 + x^2 e_2)$.

Properties:

- Straight lines through the origin in $B_r(0)$ are geodesics.
- The induced metric is Euclidean at the origin in $B_r(0)$.
- The Christoffel symbols vanish at the origin in $B_r(0)$.

$$g_{ij}(x) = \delta_{ij} + \mathcal{O}(\|x\|^2) \qquad x \in B_r(0)$$

Local Rigidity

We can thus find coordinates that make the induced metric Euclidean to first order at any point.

Question: Can we do better?

- For instance, can we achieve the ultimate simplification — can we make the metric Euclidean in an entire neighbourhood?
- Or how about just Euclidean to second order at any point?

NO! A fundamental fact is

- The equations we'd have to solve to achieve a Euclidean metric to more than second order are overdetermined.
- There are integrability conditions that have to hold:

$$0 = \frac{\partial \Gamma^s_{jk}}{\partial x^i} - \frac{\partial \Gamma^s_{ik}}{\partial x^j} + \Gamma^t_{jk}\Gamma^s_{it} - \Gamma^t_{ik}\Gamma^s_{jt} \quad \text{for all } i, j, k, s$$

Gauss' Totally Awesome Theorem

We can interpret the integrability condition in terms of curvature.

- Define the Riemann curvature (3,1)-tensor of S by

$$\mathrm{Rm}(X, Y, Z) := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$$

- Thus we can expand $\mathrm{Rm} = \sum_{ijks} R_{ijk}{}^{s}\, \omega^i \otimes \omega^j \otimes \omega^k \otimes E_s$ where

$$R_{ijk}{}^{s} = \frac{\partial \Gamma^s_{jk}}{\partial x^i} - \frac{\partial \Gamma^s_{ik}}{\partial x^j} + \Gamma^t_{jk}\Gamma^s_{it} - \Gamma^t_{ik}\Gamma^s_{jt}$$

- Now we have the Theorema Egregium of Gauss that relates the Riemann curvature tensor to the second fundamental form:

$$\boxed{R_{ijk}{}^{s} + \left(A_{jk}A^s_i - A_{ik}A^s_j\right) = 0 \qquad \text{where } A^s_i = \sum_t g^{st} A_{it}}$$

Interpretation

Let $R_{ijk\ell} := \sum_s g_{\ell s} R_{ijk}{}^s$ be the Riemann curvature (4,0)-tensor.

In two dimensions, the Theorema Egregium shows that the only independent term in $R_{ijk\ell}$ is

$$R_{1212} = -\big(\underbrace{A_{11}A_{22} - A_{12}^2}_{\text{Determinant of } A}\big)$$

The determinant of A (in an ONB) is the product of the principal curvatures, also known as the Gauss curvature!

It's an intrinsic quantity!

Isometries

Def: Surfaces S and S' with metrics g and g' are isometric if there exists $\phi : S \to S'$ s.t. for all $X_p, Y_p \in T_pS$ and all $p \in S$ we have

$$g'(D\phi(X_p), D\phi(Y_p)) = g(X_p, Y_p)\,.$$

I.e. the intrinsic geometry is preserved at corresponding points.

Examples:

- Isometries induced from rigid motions of $\mathbb{R}^3$.
- Purely intrinsic isometries.
  - $\to$ Non-planar developable surfaces.
  - $\to$ Catenoid and helicoid.
  - $\to$ Amphora and inverted amphora.
  - $\to$ Infinitesimal isometries and Killing vector fields.

The Catenoid and the Helicoid Are Isometric

Rigidity

Isometries are rare.

Fact: Curvature is a local invariant under isometry.

- The key obstruction to the existence of local isometries.
- I.e. surfaces with different curvatures can't be isometric.
- But surfaces with the same curvature are so — locally.
- Example: surfaces of constant curvature.
  - $\rightarrow$ The exponential maps can be used for this purpose.
  - $\rightarrow$ Choose a basis for T_pM and T_qN.
  - $\rightarrow$ Now consider $\exp_q^N \circ \left(\exp_p^M\right)^{-1}$.

Globally, it's more complicated!

Gauss-Bonnet Theorem

The Gauss-Bonnet Theorem shows that curvature is also a global invariant with a connection to topological type.

Theorem: Let S be a regular, oriented surface with piecewise-smooth boundary consisting of consecutive curves $C_1, \ldots, C_n$.

Let θ_i be the external angle at the $C_i \to C_{i+1}$ transition.

Then the Gauss-Bonnet formula holds:

$$\sum_i \int_{C_i} k_{C_i}(s)ds + \sum_i \theta_i + \int_S KdA = 2\pi\chi(S)$$

where k_C is the geodesic curvature of C and K is the Gauss curvature of S and $\chi(S)$ is the Euler characteristic of S.

Sketch of the Proof

- Carve S up into small triangular patches, each topologically equivalent to a disk.
- Apply the local Gauss-Bonnet theorem to each patch, and add up all contributions appropriately.
- The local Gauss-Bonnet theorem itself has a number of steps.
  1. Introduce an orthogonal coordinate system.
  2. Define the angle ϕ between vector fields V, W along a curve γ.
  3. Relate ϕ' to the covariant derivatives of V, W along γ.
  4. Let $V = \gamma'$ and W be a coordinate vector field. Relate $\vec{k}_\gamma$ to ϕ'.
  5. Integrate this relationship along γ and apply Green's Theorem.
  6. Apply the theorem of turning tangents.