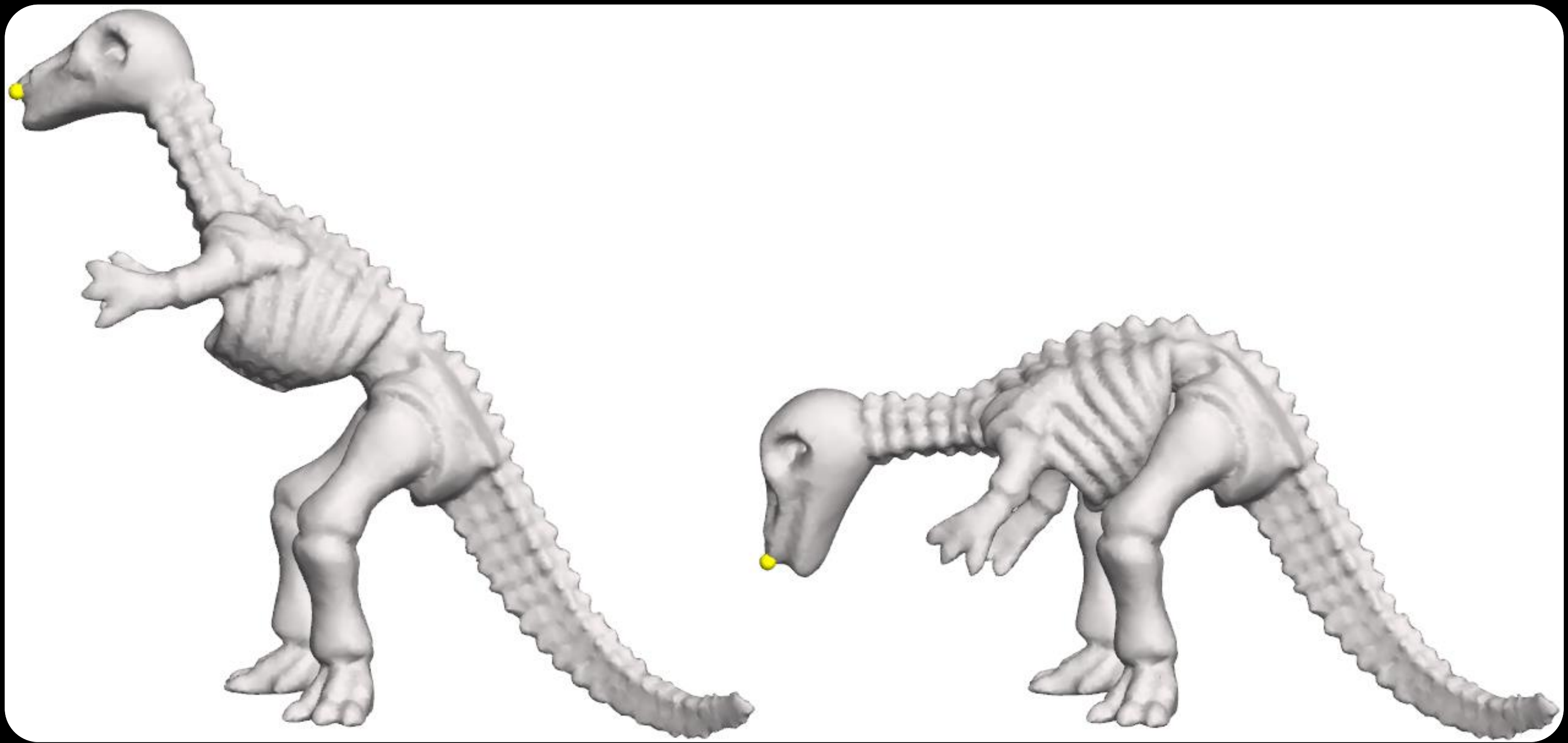
http://igl.ethz.ch/projects/ARAP/arap_web.pdf



Surface Deformation Techniques

CS 468, Spring 2013
Differential Geometry for Computer Science

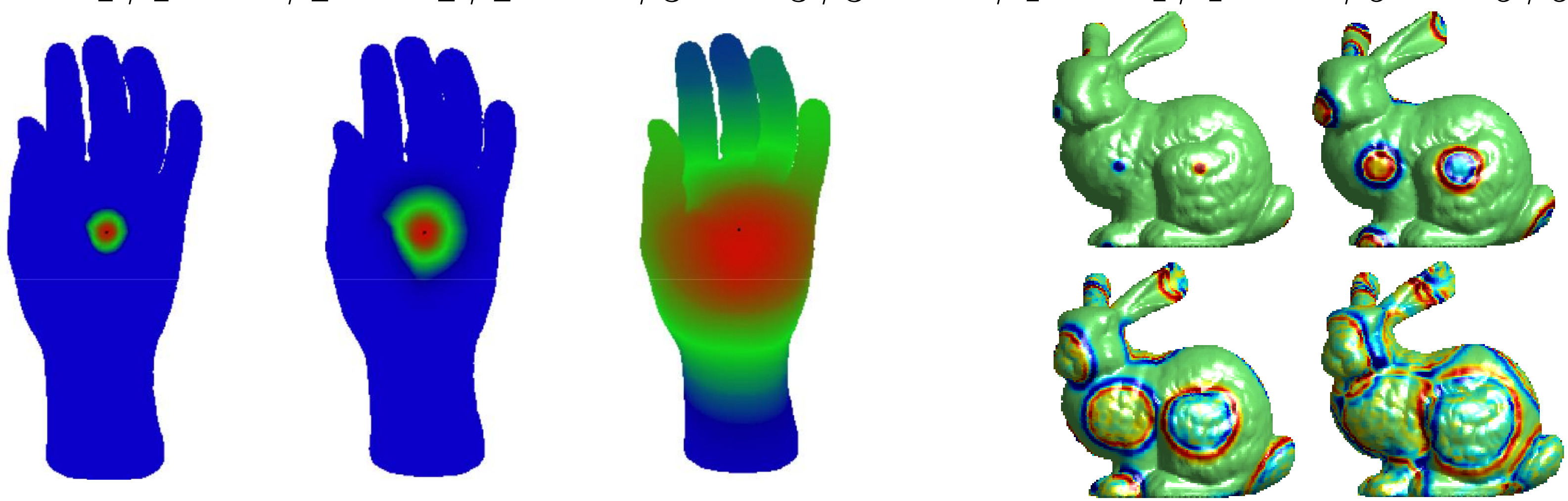
Justin Solomon and Adrian Butscher

End of Course Approaching!

- **Homework 4**: June 5
- **Project**: June 6
- **Scribe notes**: One week after, June 6 at latest
- **Course reviews**

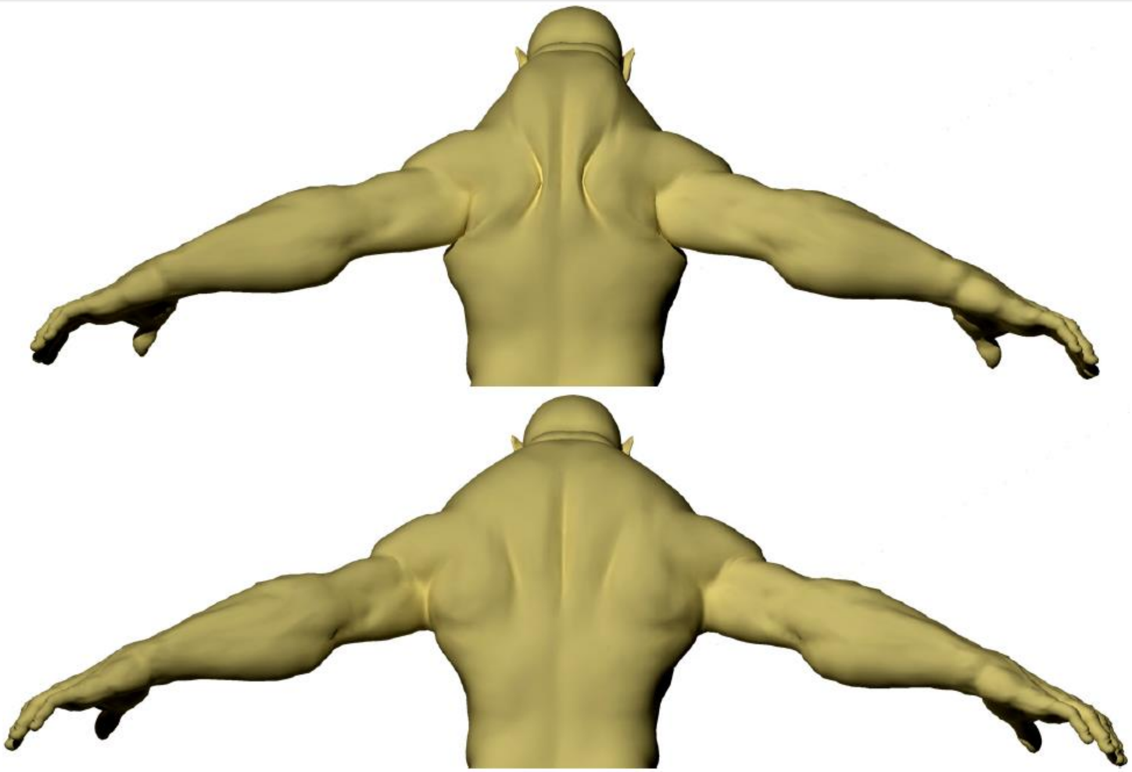
Until Now

$\Delta\phi_1 = \lambda_1\phi_1$ $\Delta\phi_2 = \lambda_2\phi_2$ $\Delta\phi_3 = \lambda_3\phi_3$ $\Delta\phi_4 = \lambda_4\phi_4$ $\Delta\phi_5 = \lambda_5\phi_5$ ...



Mostly static geometry

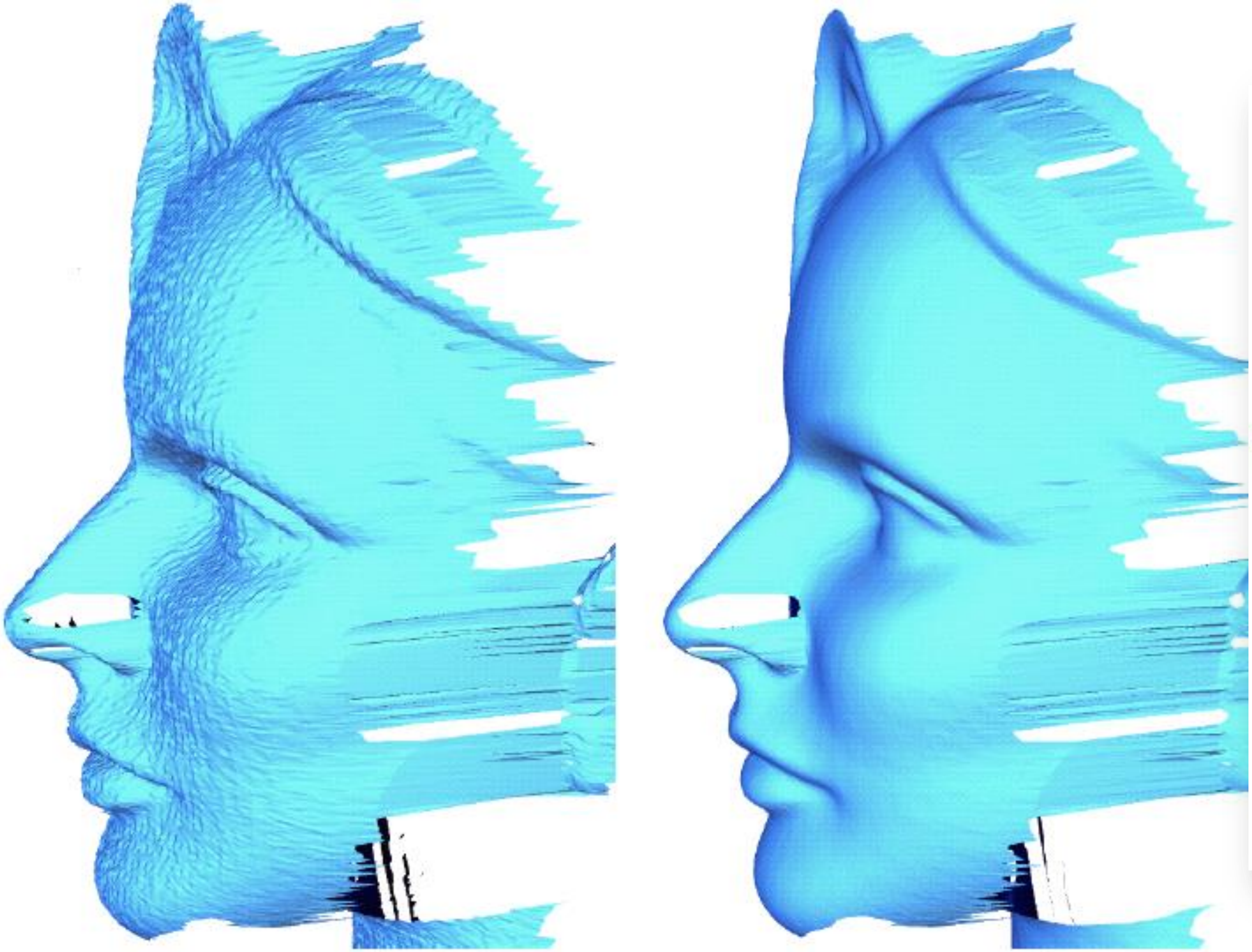
Designing Flexible Surfaces



http://igl.ethz.ch/projects/skinning/context-aware-deformation/context-aware-skinning-def.pdf

Articulation, skinning, etc. are difficult and repetitive

Less Obvious: Smoothing



http://multires.caltech.edu/pubs/ImplicitFairing.pdf

Recall:

Theoretical Viewpoint

- **Abstract surface**

  Topology plus distances

- **Embedding**

  How the surface sits in R^3

Change one, the other, or both

Most Common Technique

Modify embedding directly but characterize effect on intrinsic structure.

Near-isometry, conformality, elastic…

Example: Mean Curvature Flow

$$\frac{d\phi_t}{dt} = H_t N_t = -\Delta_t \vec{x}_t$$

↓

$$\frac{\vec{x}_{i+1} - \vec{x}_i}{\delta t} = -\Delta_i \vec{x}_{i+i}$$

Homework 3

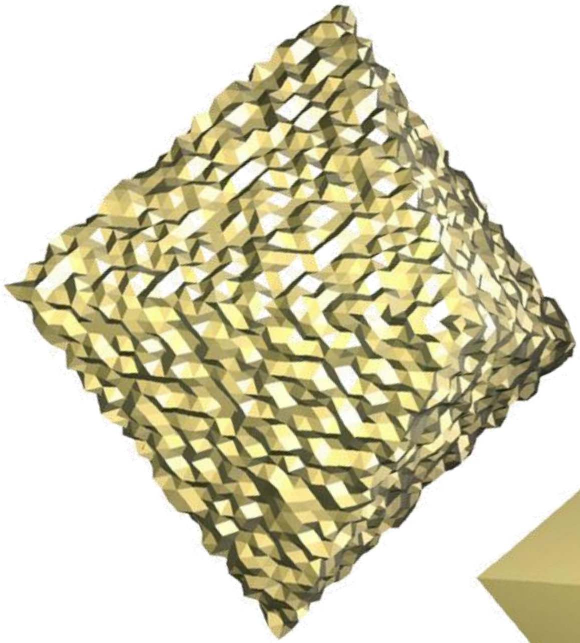
"Implicit Fairing of Irregular Meshes"
Desbrun, Meyer, Schroder, Barr; SIGGRAPH '99

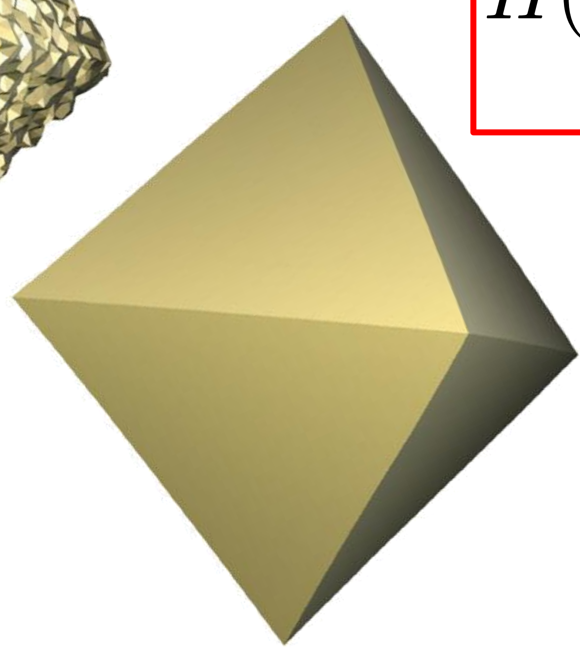
Change embedding

Variation: Anisotropic MC Flow

$$w_{\lambda,r}(H) \begin{cases} 1, & \text{for } |H| \cdot \ \lambda \\ \frac{\lambda^2}{r(\lambda-|H|)^2+\lambda^2}, & \text{for } |H| > \lambda \end{cases}$$

$$\vec{H}(p) = \frac{1}{2}\sum_{e=pq} w(H_e)H_e\vec{N}_e$$





"Anisotropic Filtering of Non-Linear Surface Features"
Hildebrandt, Polthier
Eurographics 2004

Explicit Integration Strategy

```
AnisotropicSmoothing (M, λ, s, n)
  for (steps=1... n)
    Δ_λ = 0
    for each edge e = (v_i, v_j)
      compute H_e, N_e
      Δ_λ[v_i] -= (w_λ(H_e) H_e) * N_e
      Δ_λ[v_j] -= (w_λ(H_e) H_e) * N_e
    for each triangle t = (v_i, v_j, v_k)
      compute area_t
      areaStar[v_i] += area_t
      areaStar[v_j] += area_t
      areaStar[v_k] += area_t
    for each vertex v
      v += 3s/(2areaStar[v]) * Δ_λ[v]
  return M
```

Weighted Laplacian

Lumped area weights

Explicit time step

Prescribed MC Flow

By definition, mean curvature flow wants to decrease surface area.

Try to smooth rather than reduce mean curvature.

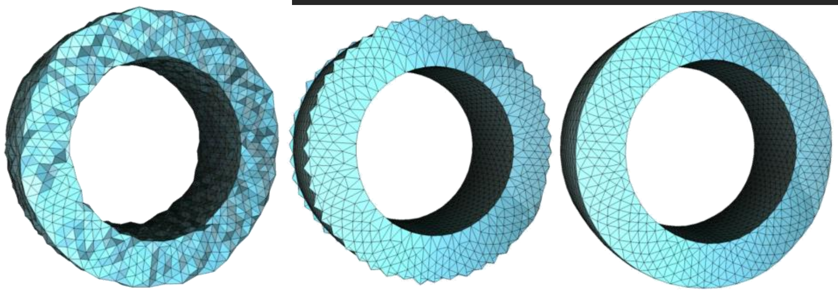
PMC Flow: Idea

$$\frac{d\mathcal{P}}{dt} = -M^{-1}(\vec{H}(\mathcal{P}) - f(\mathcal{P}) \cdot \vec{V}(\mathcal{P}))$$

MC normal from cotangent Laplacian

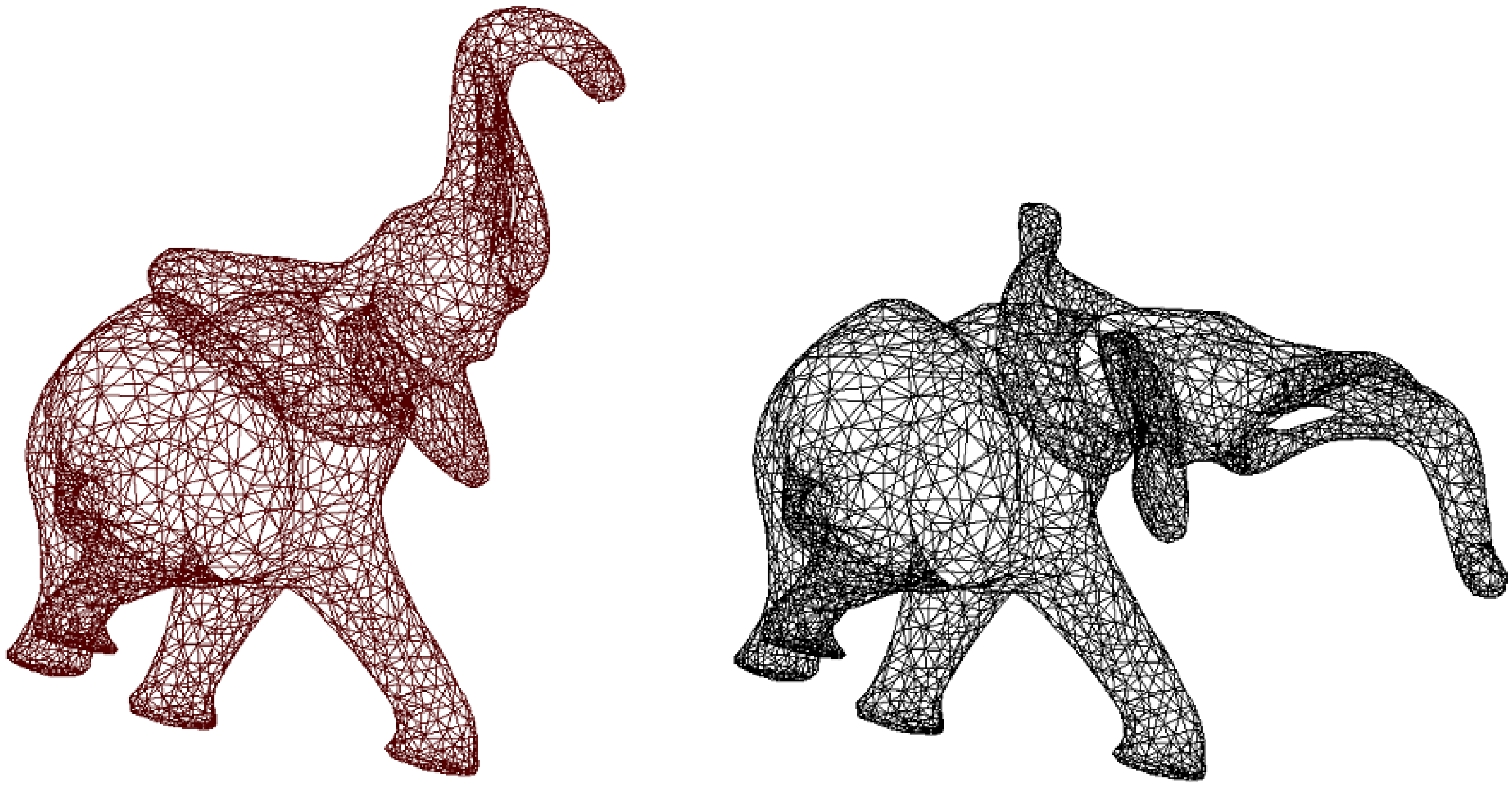
Smoothed MC target function

Gradient of volume



"Anisotropic Filtering of Non-Linear Surface Features" Hildebrandt, Polthier; EG 2004

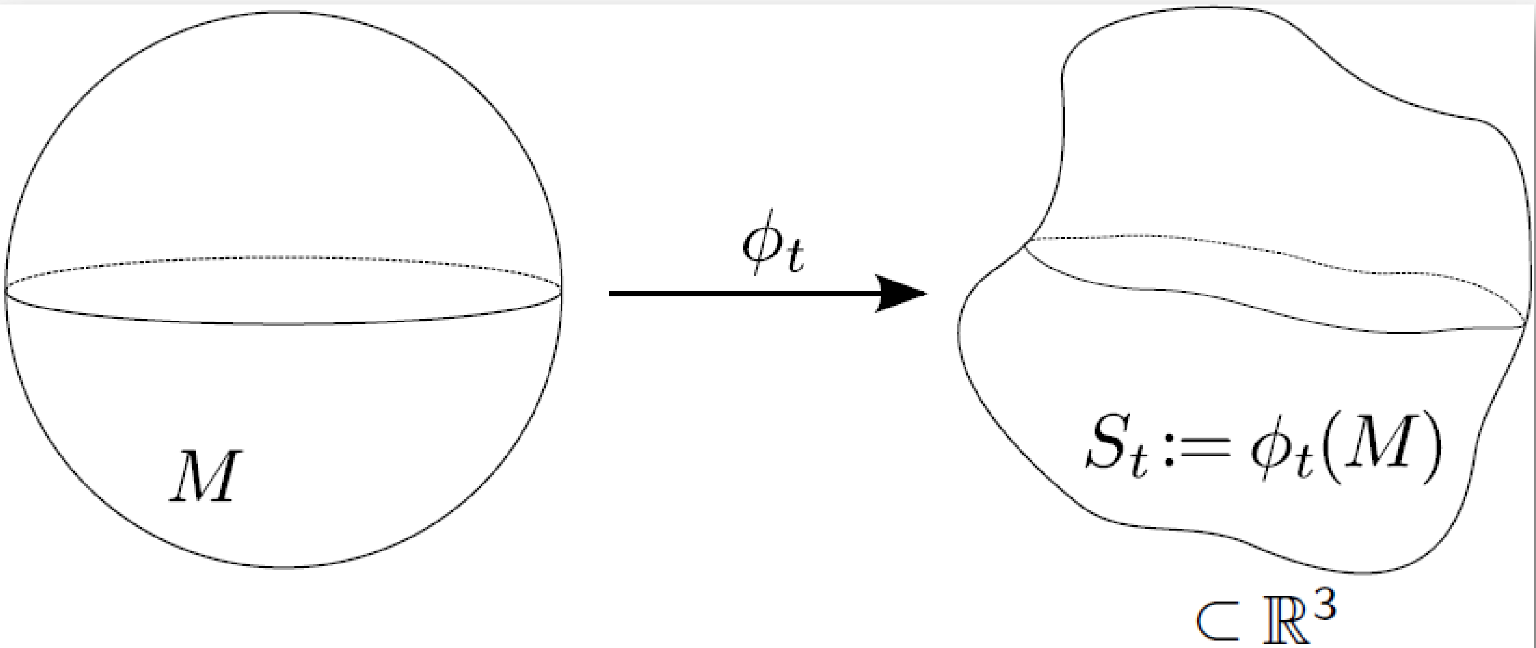
Different Example of Deformation



http://geometryfactory.com/wp/wp-content/uploads/2011/12/deformation.png

Surface editing

Elasticity: Geometric View



δ = Euclidean metric

$$g_{original} = \delta$$

$$g_{deformed} = \phi_t^* \delta := D\phi_t^\top D\phi_t$$

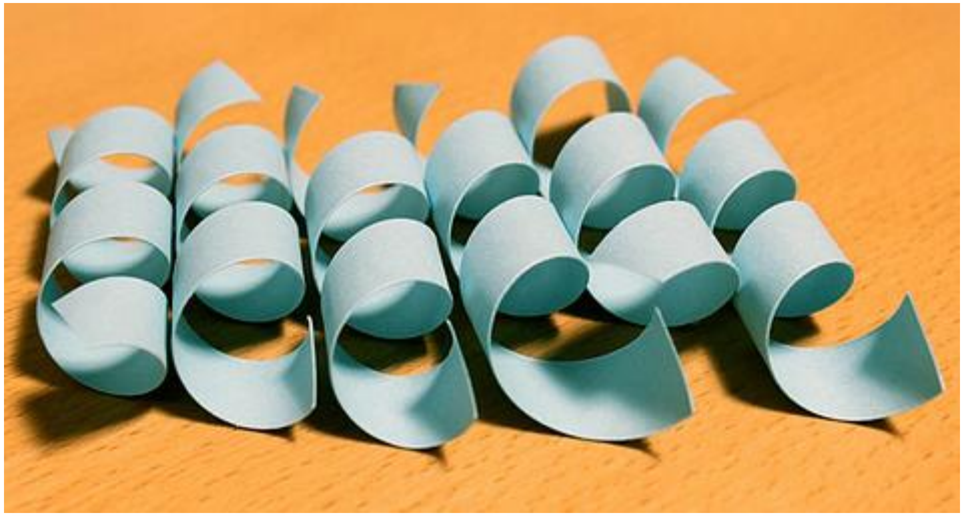
i.e. the pullback of δ under ϕ_t

Variational Point of View

$$J(\phi) \equiv \int_S W(x, E(x))\, dx$$

Energy measures deformation

Common Energy Terms



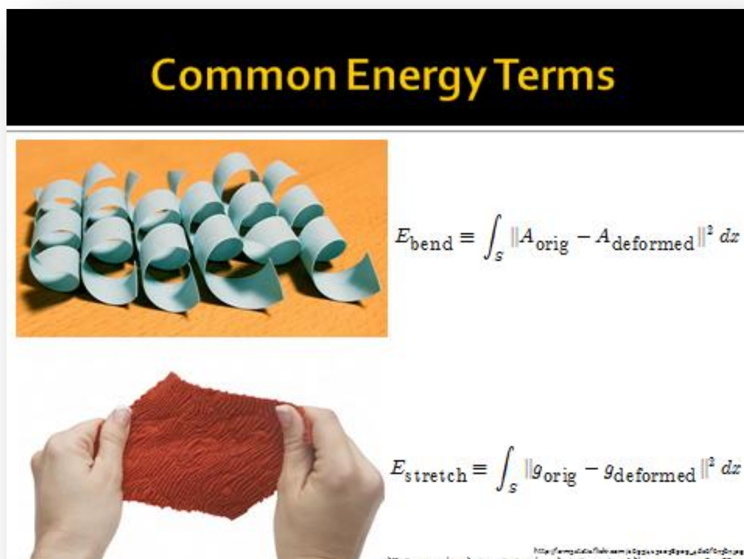
$$E_{\text{bend}} \equiv \int_S \|A_{\text{orig}} - A_{\text{deformed}}\|^2 \, dx$$

$$E_{\text{stretch}} \equiv \int_S \|g_{\text{orig}} - g_{\text{deformed}}\|^2 \, dx$$

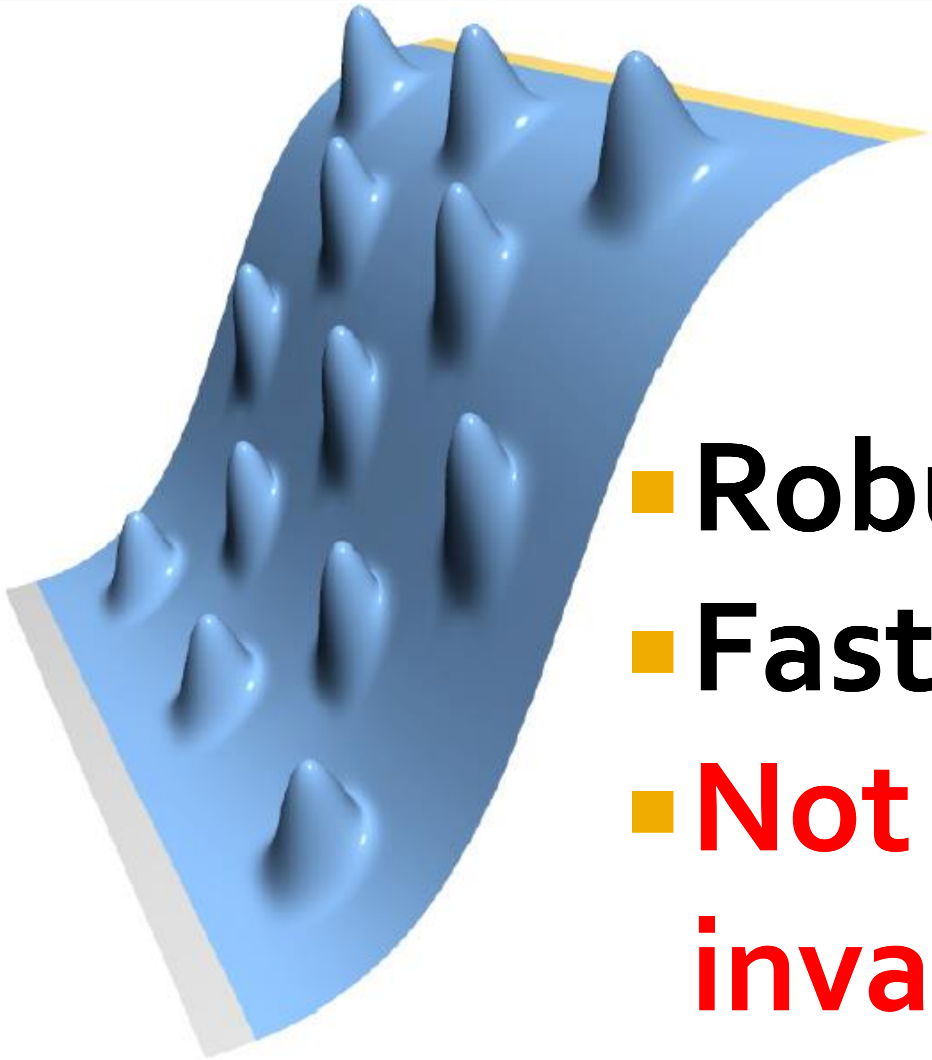
http://farm3.static.flickr.com/2693/4132058909_4dc6f675b7.jpg
http://www.popularmechanics.com/cm/popularmechanics/images/inventables-15-470-0109-39874288.jpg

Problem

Deformation energies are *highly* nonlinear.



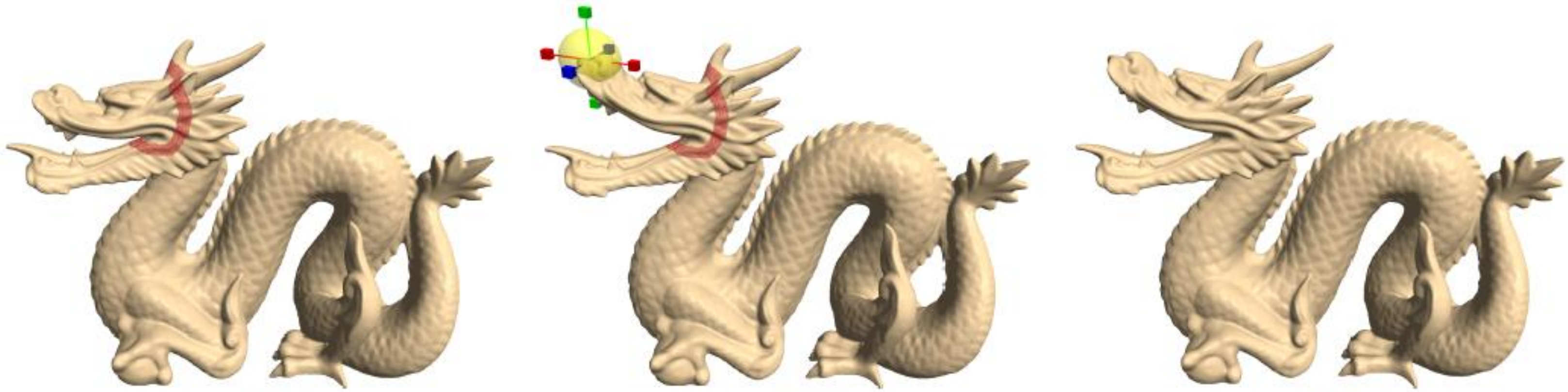
Result of Linearization



- Robust
- Fast
- Not rotation-invariant!

http://graphics.ethz.ch/Downloads/Publications/Papers/2008/Bot08/Bot08.pdf

Linear Approach



(a) (b) (c)

Figure 3: The editing process. (a) The user selects the region of interest – the upper lip of the dragon, bounded by the belt of stationary anchors (in red). (b) The chosen handle (enclosed by the yellow sphere) is manipulated by the user: translated and rotated. (c) The editing result.

"Laplacian Surface Editing"
Sorkine et al.
SGP 2004

Laplacian Coordinates

$$\delta \equiv LV = (I - D^{-1}A)V$$

Poisson equation for *V*
Approximation of normal

$$\text{minimize} \sum_i \|\delta_i - L(v_i')\|^2 + \sum_{i \in C} \|v_i' - u_i\|^2$$

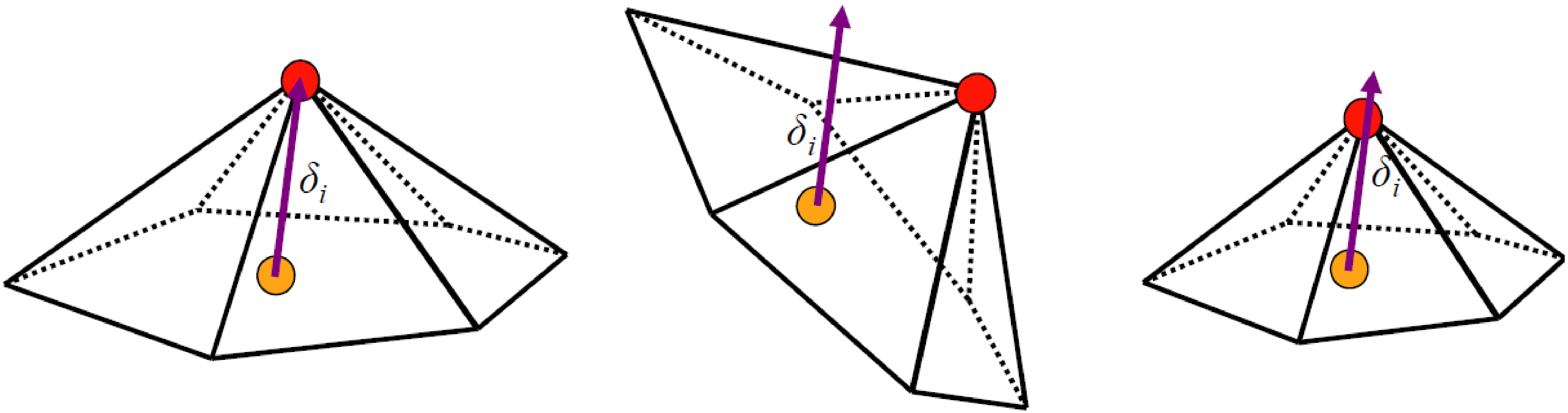
Least squares

Preserve coordinates

Respect user constraints

Translation-invariant coordinates

Problem



http://graphics.stanford.edu/courses/cs468-12-spring/LectureSlides/15_Deformation.pdf

Normal preservation is not rotation or scale invariant

Resolution

$$\text{minimize} \sum_i \|T_i \delta_i - L(v_i')\|^2 + \sum_{i \in C} \|v_i' - u_i\|^2$$

New variable

Trick: Restrict T to set of desirable transformations (or approximation thereof)

$$M_{sim} = s(\alpha I + \beta H + \gamma \vec{h}^\top \vec{h})$$

Transform source δ with vertices

Linear Approximation

$$\text{minimize} \sum_i \|T_i \delta_i - L(v_i')\|^2 + \sum_{i \in C} \|v_i' - u_i\|^2$$

New variable

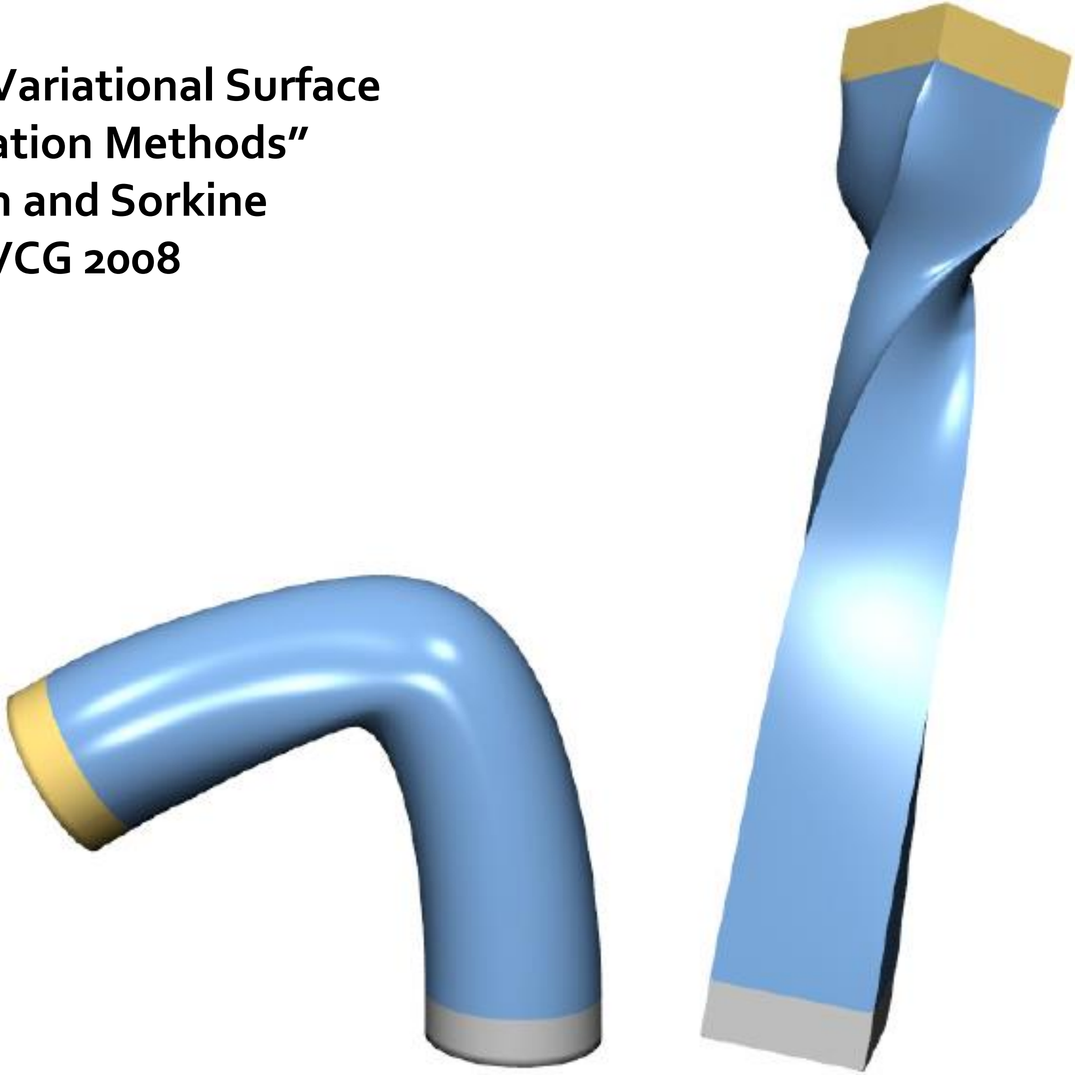
Trick: Restrict T to set of desirable transformations (or approximation thereof)

$$M_{sim} = s(\alpha I + \beta H + \gamma \bar{h}^\top \bar{h})$$

Transform source δ with vertices

Limits of Approximation

"On Linear Variational Surface Deformation Methods"
Botsch and Sorkine
TVCG 2008



http://graphics.ethz.ch/Downloads/Publications/Papers/2008/Bot08/Bot08.pdf

Nonlinear Approach

To appear at the Eurographics Symposium on Geometry Processing (2007)
Alexander Belyaev, Michael Garland (Editors)

As-Rigid-As-Possible Surface Modeling

Olga Sorkine and Marc Alexa

TU Berlin, Germany

Abstract
Modeling tasks, such as surface deformation and editing, can be analyzed by observing the local behavior of the surface. We argue that defining a modeling operation by asking for rigidity of the local transformations is useful in various settings. Such formulation leads to a non-linear, yet conceptually simple energy formulation, which is to be minimized by the deformed surface under particular modeling constraints. We devise a simple iterative mesh editing scheme based on this principle, that leads to detail-preserving and intuitive deformations. Our algorithm is effective and notably easy to implement, making it attractive for practical modeling applications.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling – geometric algorithms, languages, and systems

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As-Rigid-As-Possible

Want:

For each cell C_i around vertex *i*

$$p'_i - p'_j = R_i(p_i - p_j)\ \forall j \in \mathcal{N}(i)$$

Rotation — Center — Neighbor

Try to enforce local isometry

ARAP Cell Energy

Pointwise Laplacian

$$E(\mathcal{C}_i, \mathcal{C}_i') = \sum_{j \in \mathcal{N}(i)} w_{ij} \|(p_i' - p_j') - R_i(p_i - p_j)\|^2$$

$$E(\mathcal{S}') = \sum_i w_i E(\mathcal{C}_i, \mathcal{C}_i')$$

Area weight

ARAP Cell Energy

Pointwise Laplacian

$$E(\mathcal{C}_i, \mathcal{C}_i') = \sum_{j \in \mathcal{N}(i)} w_{ij} \|(p_i' - p_j') - R_i(p_i - p_j)\|^2$$

$$E(\mathcal{S}') = \sum_i w_i E(\mathcal{C}_i, \mathcal{C}_i')$$

Area weight

Still nonlinear!

Alternating Approach

$$E(\mathcal{C}_i, \mathcal{C}_i') = \sum_{j \in \mathcal{N}(i)} w_{ij} \|(p_i' - p_j') - R_i(p_i - p_j)\|^2$$

$$E(\mathcal{S}') = \sum_i w_i E(\mathcal{C}_i, \mathcal{C}_i')$$

Poisson equation!

Vertex positions: Linear

Alternating Approach

$$\arg\min_{R_i} \sum_{j \in \mathcal{N}(i)} w_{ij} \|e'_{ij} - R_i e_{ij}\|^2 = \arg\max_{R_i} \sum_{j \in \mathcal{N}(i)} w_{ij} e'^{\top}_{ij} R_i e_{ij}$$

$$= \arg\max_{R_i} \mathrm{Tr}\left(R_i \sum_{j \in \mathcal{N}(i)} w_{ij} e_{ij} e'^{\top}_{ij} \right)$$

$$= \arg\max_{R_i} \mathrm{Tr}\left(R_i S_i \right)$$

$$S_i = U \Sigma V^{\top}$$

$$\mathrm{Tr}(R_i S_i) = \mathrm{Tr}(R_i U \Sigma V^{\top}) = \mathrm{Tr}(V^{\top} R_i U \Sigma)$$

$$= \mathrm{Tr}(\tilde{R}\Sigma) = \mathrm{diag}\ \tilde{R} \cdot \mathrm{diag}\ \Sigma$$

$$\cdot \quad \sum \sigma_i \text{ for } \tilde{R} = I$$

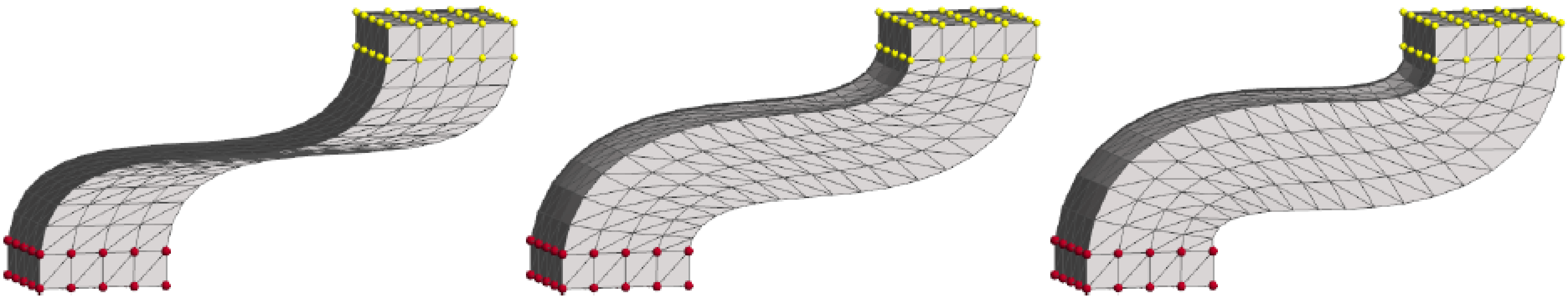
$$\tilde{R} = I \Rightarrow R_i = V U^{\top}$$

Rotation matrices: SVD

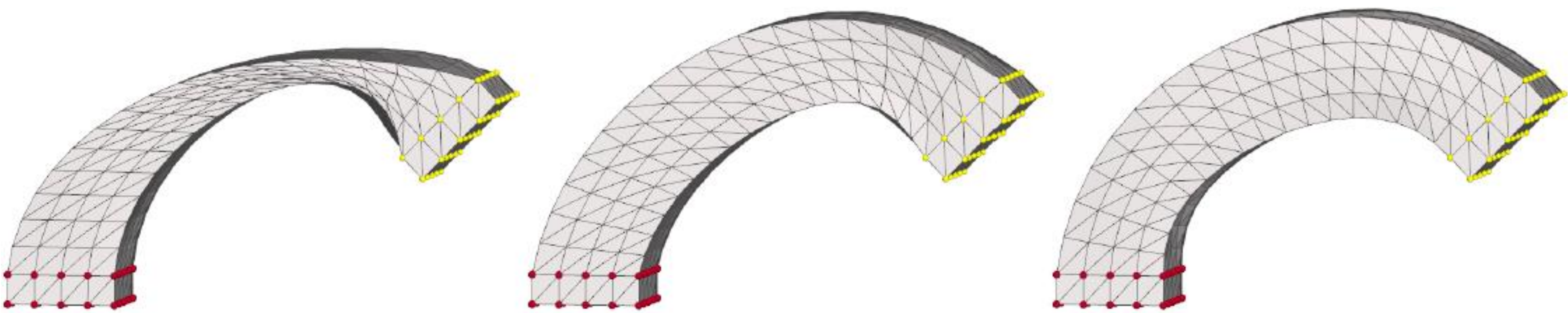
Modeling

- Vertices fixed by user
- Initialize optimization with simpler method
- Alternating optimization

Results



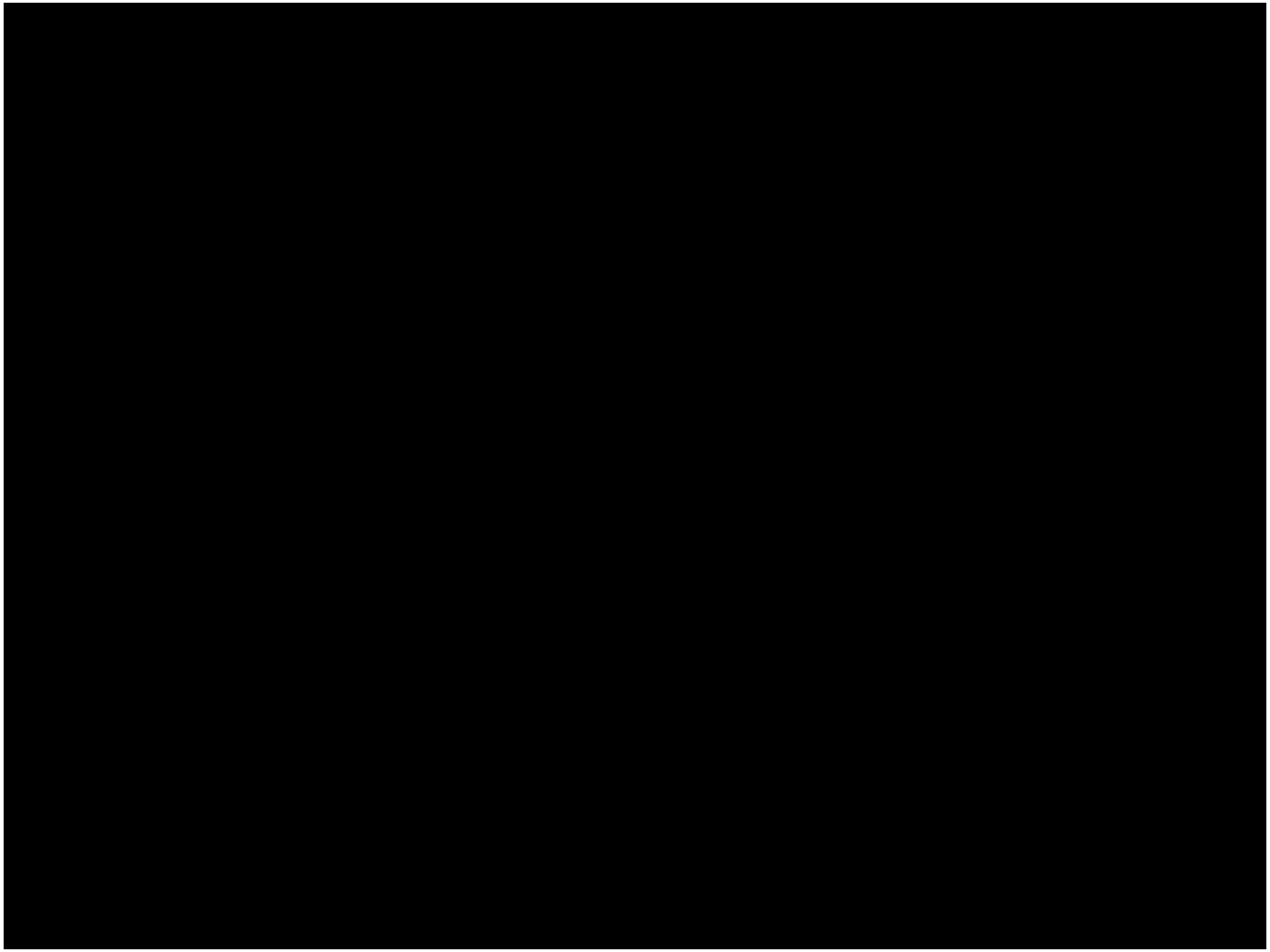
initial guess | 1 iteration | 2 iterations



initial guess | 1 iteration | 4 iterations

http://igl.ethz.ch/projects/ARAP/arap_web.pdf

Results



https://www.youtube.com/watch?v=ltX-qUjbkdc

Willmore Energy

$$\mathcal{W} \equiv \int_S H^2\, dA - \int_S K\, dA$$
$$= \int_S \|\Delta \vec{x}\|^2\, dA - 2\pi\chi(S)$$
$$= \frac{1}{4}\int_S (\kappa_1 - \kappa_2)^2\, dA$$

Conformal-invariant bending energy

Research Opportunity

mathoverflow | Questions | Tags | Users | Badges | Unanswered | Ask Question

Tweetable way to see that Willmore energy is Möbius invariant?

7

The Willmore energy $\mathcal{W} = \int_M H^2 dA$ differs from the functional

$$\widetilde{\mathcal{W}} = \int_M (H^2 - K) dA$$

just by a constant as one can see from the Gauss - Bonnet theorem (K here is the Gaussian curvature of M).

The expression $H^2 - K$ in $\widetilde{\mathcal{W}}$ is the half of the square of the length of *the trace-free part of the second fundamental form* which is a (pointwise) conformally invariant density of conformal weight -2, while "dA" can be seen as a density with conformal weight 2, so the entire integrand $(H^2 - K)dA$ is independent of a choice of a metric.

Thus $\widetilde{\mathcal{W}}$ is manifestly conformally, and in particular, Möbius invariant. So is $\mathcal{W}$.

(A Liouville's theorem ensures that conformal maps of $\mathbb{R}^n$, $n \geq 3$, are restrictions of Möbius transformations.)

Edit. The above is an attempt to address the original request for a "tweetable" argument.

Of course, the precise statement is that the Willmore energy is conformally invariant with respect to the conformal transformations of the *ambient* space. (Otherwise we would not be able to invoke the Liouville's theorem).

The correct definition of the Willmore energy involves an immersion $f\colon M \to \mathbb{R}^3$ and the induced conformal structure on the immersed manifold. The Dirac spheres show, in particular, that the Willmore energy does depend on the immersion.

http://mathoverflow.net/questions/99555/tweetable-way-to-see-that-willmore-energy-is-mobius-invariant

Discretization of Bending Energy

Eurographics Symposium on Geometry Processing (2006)
Konrad Polthier, Alla Sheffer (Editors)

A Quadratic Bending Model for Inextensible Surfaces

Miklós Bergou, Columbia University
Max Wardetzky, Freie Universität Berlin
David Harmon, Columbia University
Denis Zorin, New York University
Eitan Grinspun, Columbia University

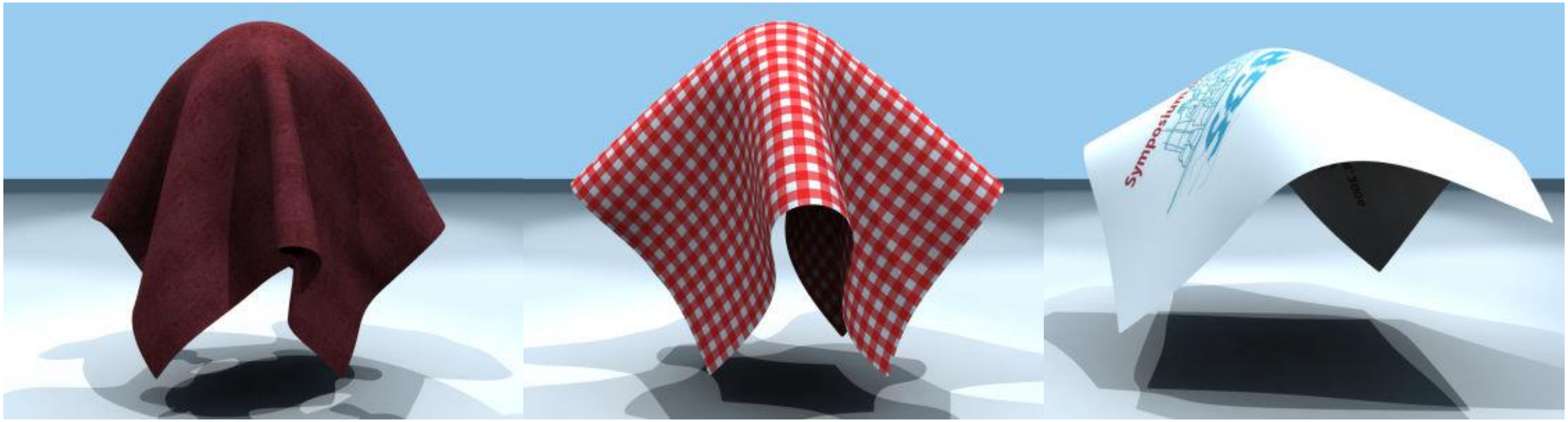
Abstract

Relating the intrinsic Laplacian to the mean curvature normal, we arrive at a model for bending of inextensible surfaces. Due to its constant Hessian, our isometric bending model reduces cloth si

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Grap
and Object Modeling

$$E_b = \int_S H^2 \, dA$$

Application in Dynamics



$$K \equiv 0$$

http://www.cs.columbia.edu/cg/pdfs/9-QuadBend.pdf

Desirable Properties

- Quadratic in vertex positions
- Invariant under isometry
  "Isometry" = preserved edge lengths
- Invariant under rigid motion
- Invariant under uniform scaling

Quadratic Energy

$$E_b = \frac{1}{2}\sum_{ij} Q_{ij}\langle \vec{x}_i, \vec{x}_j \rangle$$

- Q must be **positive semi-definite**
  Convex energy
- Q must be constructed from **intrinsic** properties
  Edge lengths, interior angles, areas, etc.
- Rows and columns **sum to 0**
  By translation invariance

Scale-Based Factorization

$$E_b = \frac{1}{2}\vec{x}_{all}^\top (L^\top M^{-1} L)\vec{x}_{all}^\top$$

Scale-independent

Scaling term

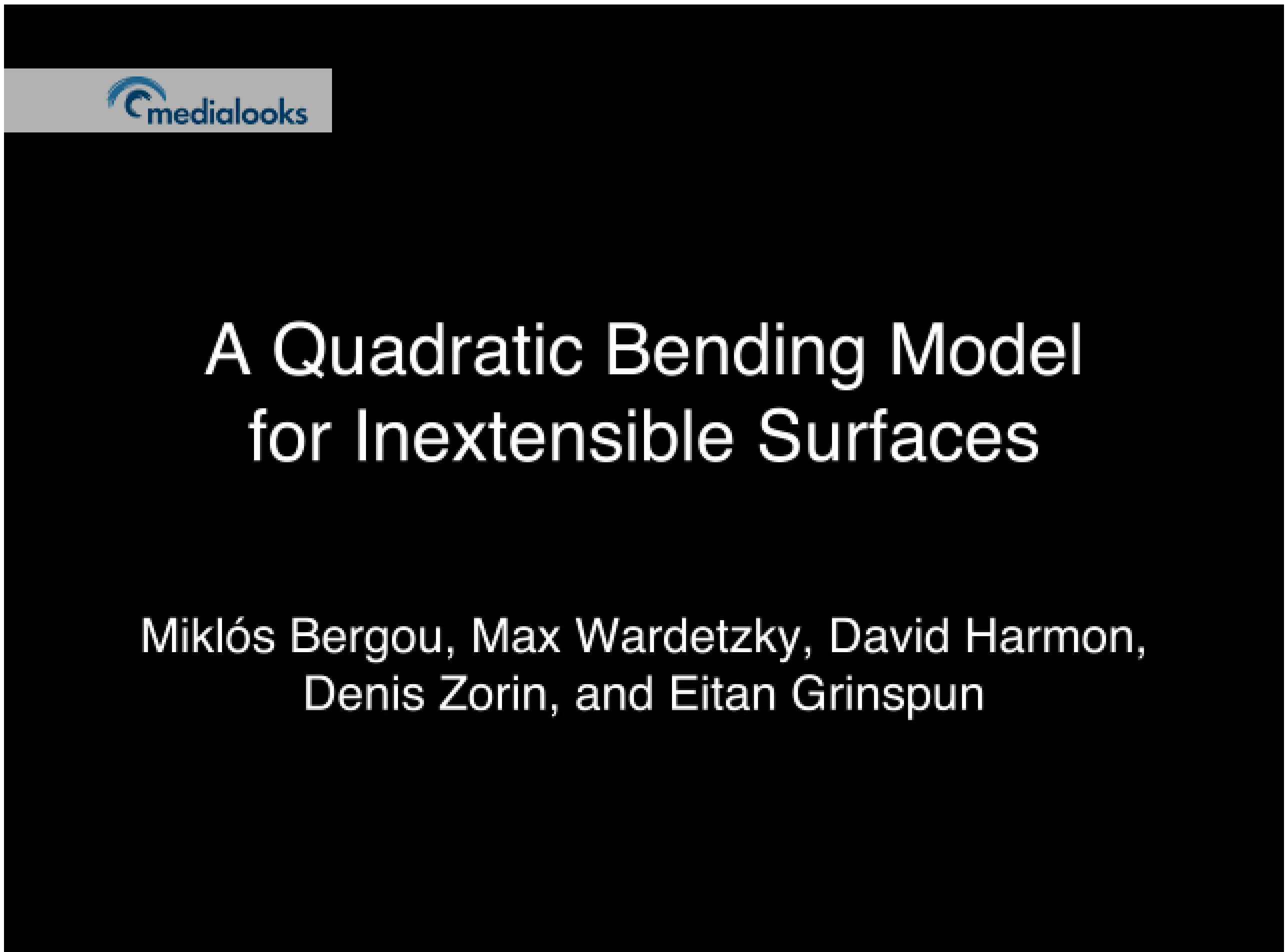
Looks like finite elements.

Observation

If the surface deforms isometrically, then Q is unchanged.

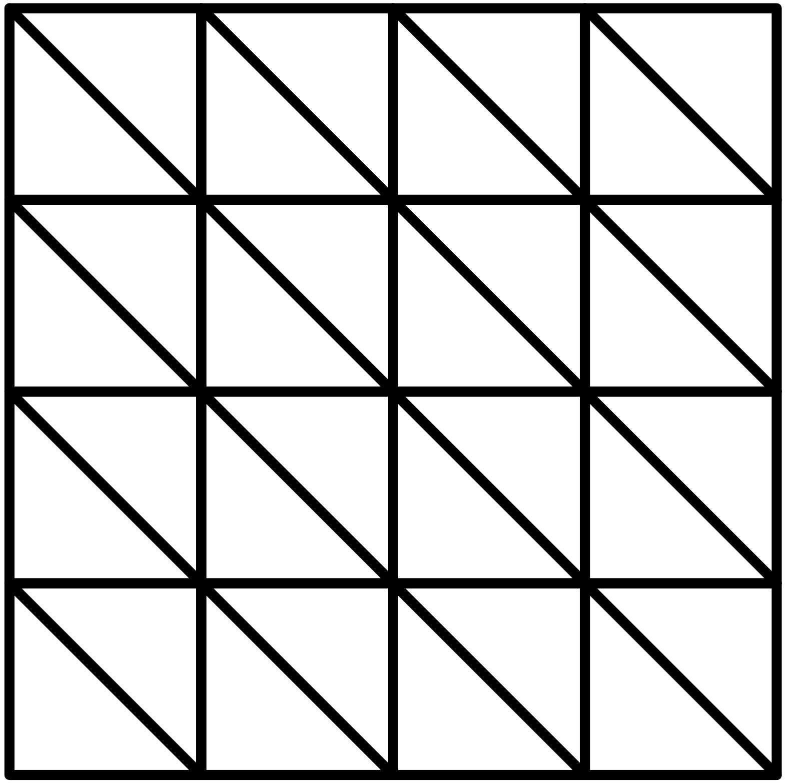
So, pre-factor the Q matrix to simplify solves.

Results



http://www.cs.columbia.edu/cg/quadratic/

Hidden Problem: Unresolved



Connection to geodesic algorithms?

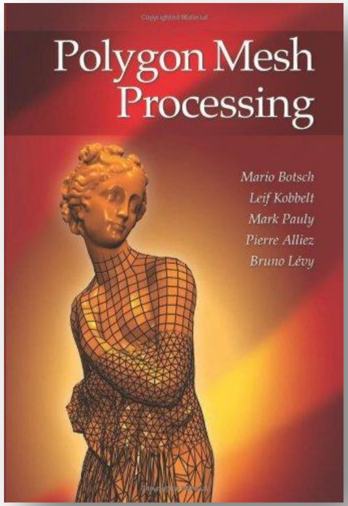
Fixing edge lengths doesn't always approximate smooth isometry.

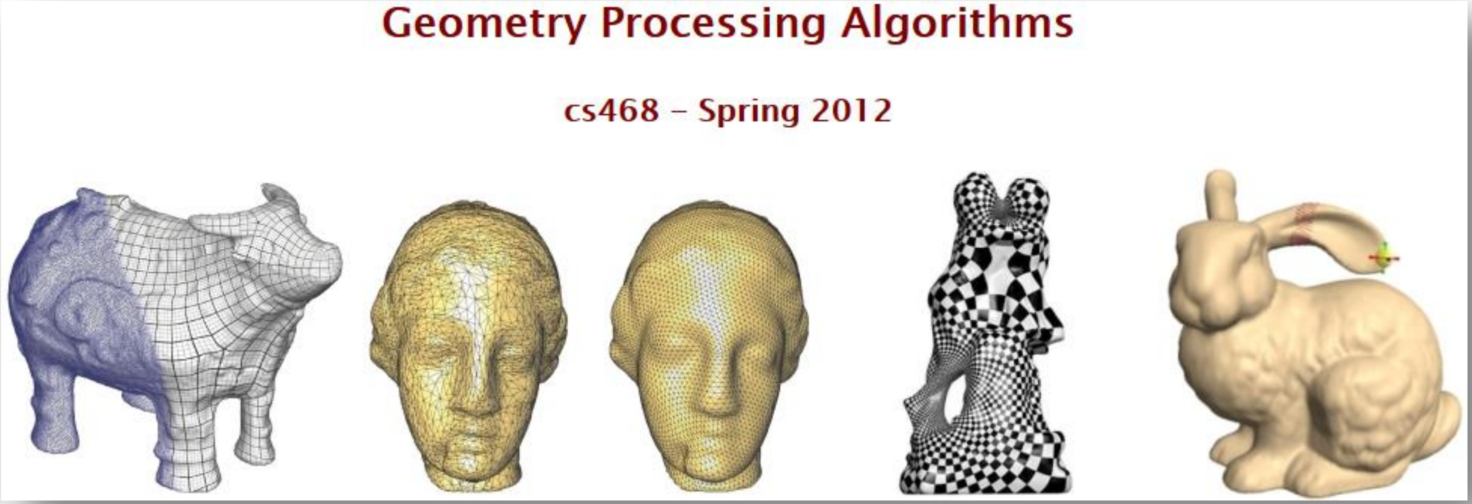
Three Ideas of Surface Deformation

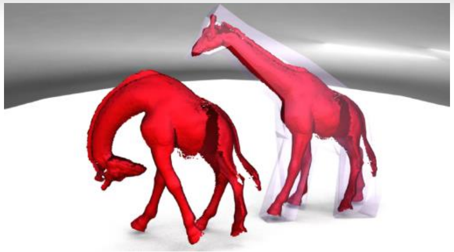
- Smoothing and fairing
- Static deformation
- Dynamic deformation

Different models and discretizations

Tip of an Iceberg

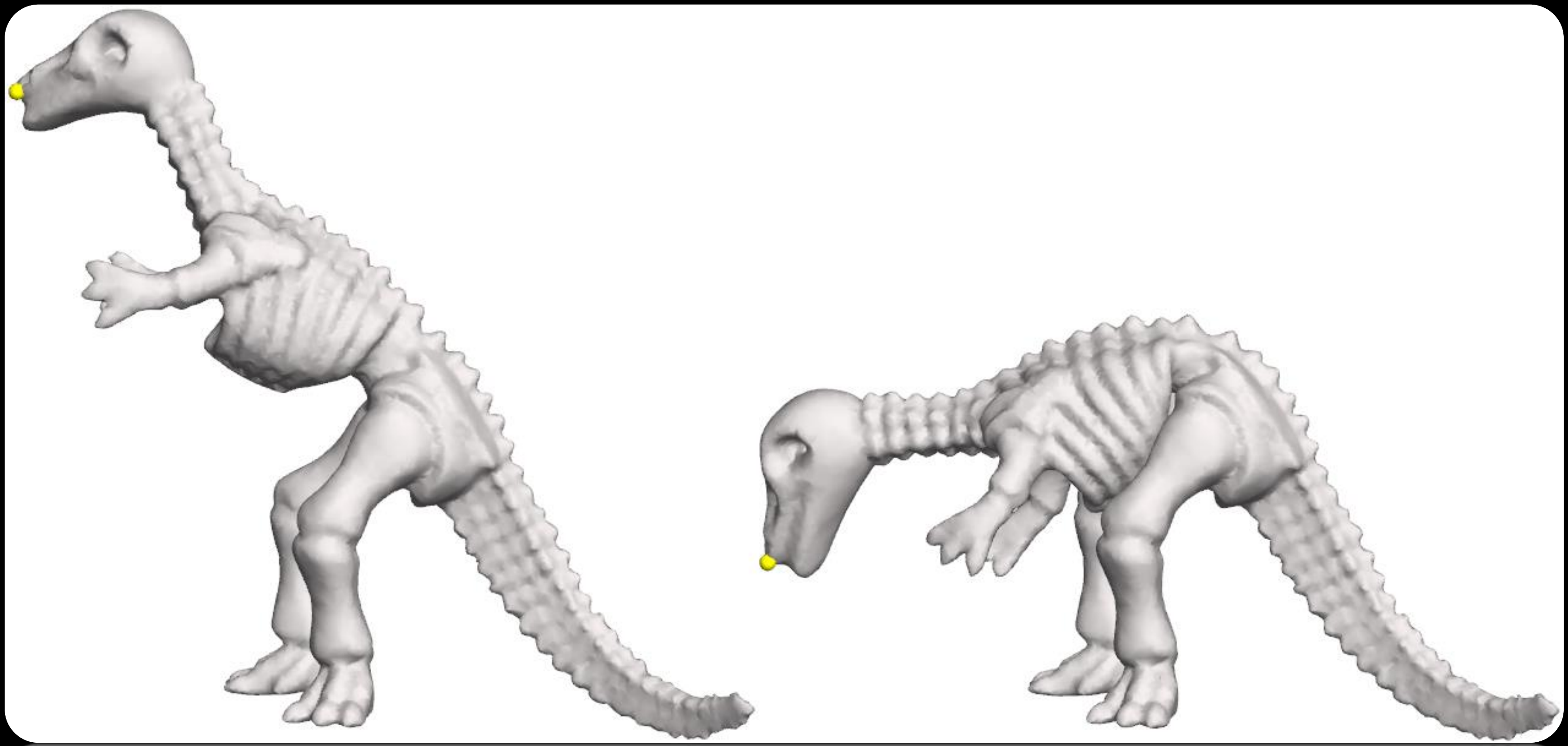






http://www.ams.org/samplings/feature-column/fcarc-harmonic
http://www.cs.technion.ac.il/~gotsman/AmendedPubl/Miri/Variational_Harmonic_Maps.pdf
http://www.stanford.edu/~justso1/assets/kvf_deformation.pdf

http://igl.ethz.ch/projects/ARAP/arap_web.pdf



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