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CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 2 Notes: Differential geometry of curves

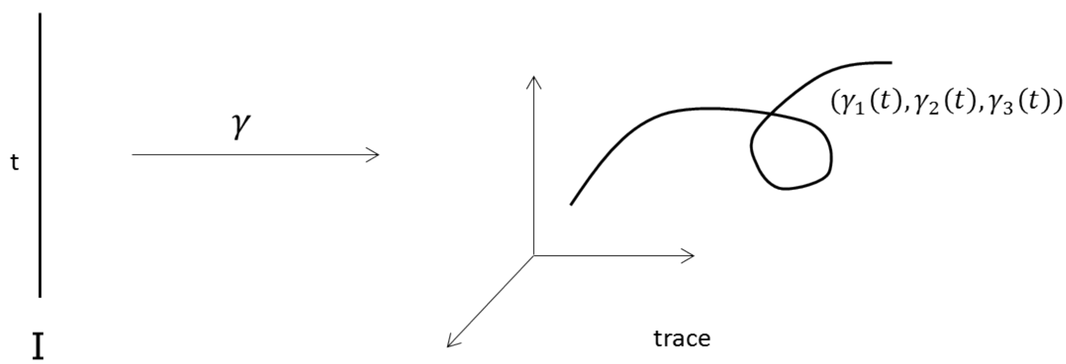
1. **Curves**

Curves are a good starting point to study surfaces as they are a little more simple (1-dimensional).

(a) *Definition*

A parametrized differentiable curve in $\mathbb{R}^n$ is a differentiable map $\gamma : I \to \mathbb{R}^n$ where $I = (a, b)$ is an open interval in $\mathbb{R}$. I can be a closed interval which gives a curve with boundary points.

Such a map has component functions $\gamma(t) := (\gamma_1(t), \ldots, \gamma_n(t))$. Each $\gamma_i : I \to \mathbb{R}$ is a differentiable function. The domain I is the space where the parameter t lives. The image of γ is the set of points $\{\gamma(t) : t \in I\} \subseteq \mathbb{R}^n$. It is a geometric thing called the *trace* of the curve. We interpret $\gamma(t)$ as the location of a particle in space at the instant of time t; and we interpret the trace of the curve as the path traced out by the particle as t varies in I.



The γ function is a mathematical way of accessing the geometry of the curve, but there is more than one access to this curve. Intuitively, one could walk at different speeds on the same trace which would lead to different parameterizations of the curve.

(b) *Velocity and acceleration*

Let's first define the terms. The instantaneous velocity of the particle at time t is the vector $\dot{\gamma}(t) = (\dot{\gamma}_1(t), \ldots, \dot{\gamma}_n(t))$. The instantaneous acceleration of the particle at time t is the vector $\ddot{\gamma}(t) = (\ddot{\gamma}_1(t), \ldots, \ddot{\gamma}_n(t))$.

Let's now look at some special curves. First, there are constant velocity curves which are straight lines. Indeed, we can write $\dot{\gamma}(t) = (c_1, c_2, c_3)$ which implies:

$$\gamma(t) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} t + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

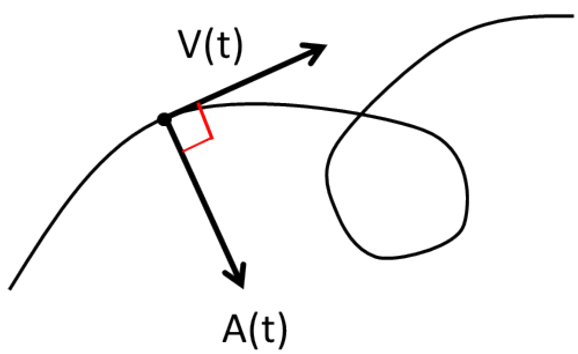
We recognize the parameterization of a line. Second, there are constant speed curves, where the acceleration is normal to the velocity. Constant speed is a weaker condition than constant velocity. It means that:

$$\|\dot{\gamma}(t)\| = (\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + \dot{\gamma}_3(t)^2)^{1/2} = C$$
$$\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + \dot{\gamma}_3(t)^2 = C^2$$

Let's differentiate both sides:

$$2\dot{\gamma}_1(t)\ddot{\gamma}_1(t) + 2\dot{\gamma}_2(t)\ddot{\gamma}_2(t) + 2\dot{\gamma}_3(t)\ddot{\gamma}_3(t) = 0$$
$$2 < \begin{pmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \\ \dot{\gamma}_3(t) \end{pmatrix}, \begin{pmatrix} \ddot{\gamma}_1(t) \\ \ddot{\gamma}_2(t) \\ \ddot{\gamma}_3(t) \end{pmatrix} >= 0$$
$$2 < \dot{\gamma}(t), \ddot{\gamma}(t) >= 0$$

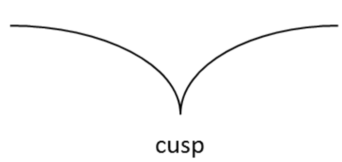
which means that the velocity and acceleration vectors are orthogonal.



(c) *Examples*

- Lines in space: $\gamma(t) = x_0 + tv$ is the line passing through x_0 in direction v.
- Circle in $\mathbb{R}^2$: $\gamma(t) = (\cos(t), \sin(t))$.
- Helix in $\mathbb{R}^3$: $\gamma(t) = (\cos(t), \sin(t), t)$

- The parameterized map can still be differentiable but the trace may not be smooth. For example: $\dot{\gamma}(t) = (t^3, t^2)$ has a cusp. It can also be the other way around: imagine a circle where one jams on the accelerator.



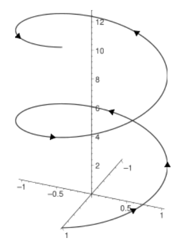


Figure 1: Helix image borrowed from http://en.wikipedia.org/wiki/Helix

(d) *Change of parameter*

A reparametrization is a bijective map $\phi : J \to I$ that gives a new curve $\tilde{\gamma} : J \to \mathbb{R}^n$ defined by $\tilde{\gamma}(s) = \gamma(\phi(s))$. The formula $t = \phi(s)$ is a *change of parameter*. A smooth mapping ϕ between intervals is a bijection if and only if ϕ' never vanishes. The trace remains unchanged but it has an effect on the velocity and the acceleration:

$$\begin{aligned}
\frac{d\tilde{\gamma}(s)}{ds} &= \frac{d\gamma(\phi(s))}{ds} = \frac{d\gamma}{dt} \circ \phi(s) \frac{d\phi(s)}{ds} \qquad \text{Note length changes} \\
\frac{d^2\tilde{\gamma}(s)}{ds^2} &= \frac{d}{ds}\left(\frac{d\gamma}{dt} \circ \phi(s) \frac{d\phi(s)}{ds}\right) \\
&= \frac{d^2\gamma}{dt^2} \circ \phi(s) \left(\frac{d\phi(s)}{ds}\right)^2 + \frac{d\gamma}{dt} \circ \phi(s) \frac{d^2\phi(s)}{ds^2}
\end{aligned}$$

2. **Arc length parameterization**

(a) *Arc length*

How could we calculate the length of a curve? By means of line segments, we could calculate a discrete approximation of the length of a differentiable curve. As the length of the line segments $\to 0$, the limit yields the arc length integral.

Let's derive this integral: let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth curve and partition $I = [t_0, t_1] \cup \cdots \cup [t_{n-1}, t_n]$ with $t_0 = a$ and $t_n = b$. Suppose $\gamma(t) = (x(t), y(t), z(t))$. Now compute:

$$\begin{aligned}
\mathit{length}(\gamma([a,b])) &\approx \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| \\
&= \sum_{i=1}^{n} \left((x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2\right)^{1/2} \\
&= \sum_{i=1}^{n} \left((\dot{x}(t_i^*)\Delta t_i)^2 + (\dot{y}(t_i^*)\Delta t_i)^2 + (\dot{z}(t_i^*)\Delta t_i)^2\right)^{1/2} \qquad \text{Mean value theorem; } t_i^* \in [t_{i-1}, t_i] \text{ and } \Delta_i := |t_i - t_{i-1}|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left((\dot{x}(t_i^*))^2 + (\dot{y}(t_i^*))^2 + (\dot{z}(t_i^*))^2\right)^{1/2} \Delta t_i \\
&\xrightarrow{n\to\infty} \int_a^b \left((\dot{x}(t))^2 + (\dot{y}(t)))^2 + (\dot{z}(t))^2\right)^{1/2} dt \\
&= \int_a^b \|\dot{\gamma}(t)\| dt
\end{aligned}
$$

But there is more we need to prove. Indeed, the length is a geometric quantity and should not depend on the parameterization, which is just a tool to manipulate the geometry of the curve. However, in the way we computed the arc length, it did depend on the parameterization. Let's now show that we would get the same result regardless of the parameterization we choose: let $\phi : [a, b] \to [a, b]$ be a diffeomorophism with $\phi(a) = a$ and $\phi(b) = b$. Let $\tilde{\gamma}(s) := \gamma(\phi(s))$. Then

$$
\begin{aligned}
length(\tilde{\gamma}([a,b])) &= \int_a^b \left\| \frac{d\gamma \circ \phi(s)}{ds} \right\| ds \\
&= \int_a^b |\phi'(s)| \left\| \frac{d\gamma}{dt} \circ \phi(s) \right\| ds \qquad \text{Let } t = \phi(s) \text{ so } dt = \phi'(s)ds \text{ and thus } ds = (\phi'(s))^{-1}dt = (\phi' \circ \phi^{-1}(t))^{-1}dt \\
&= \int_a^b |\phi' \circ \phi^{-1}(t)| \left\| \frac{d\gamma(t)}{dt} \right\| \frac{dt}{|\phi' \circ \phi^{-1}(t)|} \\
&= \int_a^b \left\| \frac{d\gamma(t)}{dt} \right\| dt \\
&= length(\gamma([a,b]))
\end{aligned}
$$

Now suppose we have a parameterization of γ such that $\|\dot{\gamma}(t)\| = 1$. Then we have:

$$
\begin{aligned}
length(\gamma([0,T])) &= \int_0^T \|\gamma(t)\| dt \\
&= \int_0^T dt \\
&= T
\end{aligned}
$$

This parameterization is called the arc-length parameterization of the curve. It can be proven that it always exists.The arc-length parametrization is very useful theoretically (as we'll see) but difficult to work with in practice because the arc-length can be hard to compute (i.e. finding the function ℓ) and it's inverse can then be very hard to find (i.e. inverting to find ℓ^{-1}).

Let's look at an example: we have $s = \sqrt{2}e^t$ for the logarithmic spiral so $t = \log(s/\sqrt{2})$. Hence the re-parametrized version of the logarithmic spiral is:

$$
\tilde{\gamma}(s) = \frac{s}{\sqrt{2}} \left(\cos(\log(s/\sqrt{2})), \sin(\log(s/\sqrt{2})) \right).
$$

(b) *Curvature*

The geodesic curvature vector can be defined in an arbitrary parametrization as the normal component of the acceleration vector, normalized by the squared length of the tangent vector:

$$\vec{k}_\gamma := \frac{1}{\|\dot{\gamma}\|^2}\left(\ddot{\gamma} - \frac{\langle \ddot{\gamma}, \dot{\gamma}\rangle}{\|\dot{\gamma}\|^2}\dot{\gamma}\right) = \frac{1}{\|\dot{\gamma}\|}\left[\frac{d}{dt}\left(\frac{\dot{\gamma}}{\|\dot{\gamma}\|}\right)\right]^\perp$$

Rate of change of the unit tangent vector perpendicular to the curve

The curvature (scalar) is then defined as $k_\gamma := \|\vec{k}_\gamma\|$.

In the arc length parametrization we have $\vec{k}_\gamma = \|\ddot{\gamma}\|$ since the acceleration vector is orthogonal to the curve.

3. **Frenet frame**

(a) *Definition*

Let $\gamma :\to \mathbb{R}^3$ be a curve, without loss of generality parametrized by arc-length. We will now find a *canonical framing* of γ, namely a choice of “moving axes” (three linearly independent vectors attached to each point $\gamma(s)$) that is best adapted to its geometry.

Let $T(s) := \dot{\gamma}(s)$. Then $\|T(s)\| = 1$ for all s since γ is parametrized by arc-length. This is our first vector.

A point of non-zero curvature allows us to define a distinguished normal vector. Recall that we have $0 = \frac{d}{ds}\|\dot{\gamma}(s)\|^2 = 2\langle T(s), \dot{T}(s)\rangle = 2\langle T(s), \vec{k}_\gamma(s)\rangle$. Thus the curvature vector is normal to γ. Since it's not equal to zero, we can divide by its magnitude and obtain a unit normal vector field $N(s) := \dot{T}(s)/\|\dot{T}(s)\|$ along γ. This is our second vector in the moving axis. We also define the *osculating plane* at $\gamma(s)$ to the plane spanned by $T(s)$ and $N(s)$.

We now define the *binormal vector*, the third vector in our moving axes, to be $B(s) := T(s) \times N(s)$. This is also a unit vector and is orthogonal to both $T(s)$ and $N(s)$.

The *Frenet frame* for γ is the set of moving axes $\{T(s), N(s), B(s)\}$ and is defined at each point $\gamma(s)$ where $k_\gamma(s) \neq 0$.

(b) *Frenet formulas*

The Frenet formulas explain the variation in the Frenet frame along γ. That is, we have

$$\begin{aligned}
\dot{T}(s) &= k_\gamma(s)N(s) \\
\dot{N}(s) &= \langle \dot{N}(s), T(s)\rangle T(s) + \langle \dot{N}(s), N(s)\rangle N(s) + \langle \dot{N}(s), B(s)\rangle B(s) \\
&= -k_\gamma(s)T(s) + \langle \dot{N}(s), B(s)\rangle B(s)
\end{aligned}$$

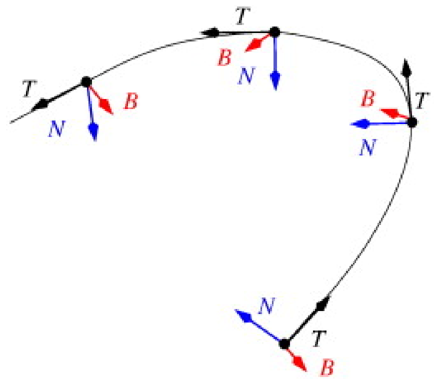


Figure 2: Frenet frame illustration borrowed from
http://www.sciencedirect.com/science/article/pii/S1053811908011981

$$
\begin{aligned}
&= -k_\gamma(s)T(s) - \tau_\gamma(s)B(s) \\
\dot{B}(s) &= \langle \dot{B}(s), T(s)\rangle T(s) + \langle \dot{B}(s), N(s)\rangle N(s) + \langle \dot{B}(s), B(s)\rangle B(s) \\
&= -\langle B(s), \dot{T}(s)\rangle T(s) + \langle \dot{B}(s), N(s)\rangle N(s) \\
&= -k_\gamma(s)\langle B(s), N(s)\rangle T(s) - \langle N(s), \dot{B}(s)\rangle B(s) \\
&= \tau_\gamma(s)B(s)
\end{aligned}
$$

Note that here we have introduced the *torsion* $\tau_\gamma(s) := -\langle \dot{N}(s), B(s)\rangle$.

(c) *Local and global theorems*

The local theorem says the following:
Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a curve with non-zero curvature. Let $k := k_\gamma(0)$ and $\tau = \tau_\gamma(0)$ and $k' = \dot{k}_\gamma(0)$. Then

$$
\begin{aligned}
\gamma(s) &\approx \gamma(0) + s\dot{\gamma}(0) + \frac{s^2}{2}\ddot{\gamma}(0) + \frac{s^3}{6}\dddot{\gamma}(0) \\
&= \left(s - \frac{k^2 s^3}{6}\right)T(0) + \left(\frac{s^2 k}{2} + \frac{s^3 k'}{6}\right)N(0) - \frac{k\tau s^3}{6}B(0)
\end{aligned}
$$

Thus locally, k and k' determine the amount of turning in the $\{T(0), N(0)\}$-plane, while τ and k determine the amount of lifting out of the $\{T(0), N(0)\}$-plane in the $B(0)$-direction.

Now the global theorem, also called the Fundamental Theorem of Curves says:

"Given differentiable functions $k : I \to \mathbb{R}$ with $k > 0$, and $\tau :\to \mathbb{R}$, there exists a regular curve $\gamma : I \to \mathbb{R}^3$ such that s is the arc-length, $k(s)$ is the geodesic curvature, and $\tau(s)$ is the torsion. Any other curve satisfying the same conditions differs from γ by a rigid motion."