CS 468

Differential Geometry for Computer Science

Lecture 4 — The Definition of a Surface

Outline

- Important background knowledge.
  - → The differential of a function.
  - → The inverse and implicit function theorems.
- Different kinds of surfaces.

The Differential of a Function

- The tangent space of $\mathbb{R}^n$ at p, denoted $T_p\mathbb{R}^n$.
- Characterization of tangent vectors as tangent vectors of curves. Given $X_p \in T_pM$ we can find $c : I \to \mathbb{R}^n$ a curve with $c(0) = p$ and $\dot{c}(0) = X_p$.
- The differential of $f : \mathbb{R}^n \to \mathbb{R}^m$ at p is the matrix

$$Df_p \in \mathbb{R}^{m\times n} \quad \text{where} \quad Df_p := \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

- Interpretation as a linear mapping $Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ via the mage of curves and their tangent vectors.

$$\frac{d}{dt} f(c(t))\Big|_{t=0} = Df_p \cdot X_p$$

The Rank of the Differential

Qualitative picture of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ of locally constant rank.

Result: We can locally "modify" f into an equivalent map $\tilde{f}$ s.t.

- Case 1: Df_p is injective for all $p \in \Omega \subseteq \mathbb{R}^n$ then $n \leq m$ and

$$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$$

- Case 2: Df_p is surjective for all $p \in \Omega$ then $n \geq m$ and

$$\tilde{f}(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = (x^1, \ldots, x^m)$$

- Case 3: Df_p is bijective for all $p \in \Omega$ $n = m$ and

$$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n)$$

- Case 4: Df_p has rank k for all $p \in \Omega$ then $k \leq \min(n, m)$ and

$$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

Summary

If Df_p has constant rank then f behaves like Df_p near p.

The Inverse and Implicit Function Theorems

Proofs of these results are based on two key technical theorems.

Inverse Function Theorem

- If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth with Df_p bijective, then f is invertible on a neighbourhood of p.
- Note that Df_p is bijective at p iff $\det(Df_p) \neq 0$.

Implicit Function Theorem

- If $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth with $D_2F_{(p,q)}$ bijective and $F(p,q) = 0$, then the equation $F(x,y) = 0$ can be solved for points (x,y) near (p,q) in the following sense.
- There exists a function $g : \mathbb{R}^k \to \mathbb{R}^n$ defined near q such that $q = g(p)$ and also $F(x, g(x)) = 0$.
- We can compute Dg_x in terms of $D_1F_{(x,g(x))}$ and $D_2F_{(x,g(x))}$.

Three Kinds of Surfaces

Common representations of surfaces in $\mathbb{R}^3$.

- Graphs of functions $f : \mathbb{R}^2 \to \mathbb{R}$.
- Level sets of functions $F : \mathbb{R}^3 \to \mathbb{R}$.
  - $\to$ Graphs as level sets
  - $\to$ Level sets as graphs — relation to the Implicit Fn. Thm.
- Parametric surfaces $\sigma : U \to \mathbb{R}^3$ where $U \subseteq \mathbb{R}^2$ is an open domain in the plane and

$$\sigma(u^1, u^2) := (\sigma^1(u^1, u^2), \sigma^2(u^1, u^2), \sigma^3(u^1, u^2))$$

  - $\to$ Useful relation with level sets: $F(\sigma(u)) = const$.