STANFORD UNIVERSITY

CS 468:: DISCRETE DIFFERENTIAL GEOMETRY

Surface Theory II

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1 Representing a Surface

A surface can be represented as a graph, inverse of a level set function as well as parametric map from an open set $\mathbf{U} \subseteq \mathbb{R}^2$ to an open set in $\mathbb{R}^3$. At the same time it is important to note that not every surface can be represented as a single graph. The parametric mapping for a given surface need not be unique as well. Here we comment on defining a surface in terms of a parametric mapping, and we connect with graphs and level set functions.

1.1 Definition of Surface

A *subset* $S \subset \mathbb{R}^3$ is a *regular* surface if, for each $\mathbf{p} \in S$ there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{X} : \mathbf{U} \to V \cap S \in \mathbb{R}^3$ such that ,

$$\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v)) \tag{1}$$

We have assumed the parametrization $\mathbf{X}$ to be regular so it satisfies following conditions:

(i) $\mathbf{X}$ has continuous partial derivatives in $\mathbf{U}$ of all orders.

(ii) $\mathbf{X}$ is homeomorphism. ($\mathbf{X}$ is smooth , one to one and $\mathbf{X}^{-1}$ is smooth as well.)

(iii) For each $\mathbf{r} \in \mathbf{U}$, the differential $d\mathbf{X}_\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The mapping $\mathbf{X}$ is referred to as the parametrization or a system of (local) coordinates in (a neighbodhoord of) $\mathbf{p}$. The neighborhood $V \cap S$ of $\mathbf{p} \in S$ is called a coordinate neighborhood. In contrast to the treatment to the curves, we have defined a surface as a subset S of $\mathbb{R}^3$, and not as a map. This is achieved by covering S with the traces of parametrization which satisfy above described conditions. The requirement of the conditions **(i)** and **(ii)** are simply understood as we would like to do some differential geometry on S as well as prevent the surface from self intersections. We will comment on the requirement of the condition **(iii)** as it will guarantee the existence of a *tangent plane* at all points on the surface. By requiring $d\mathbf{X}_\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ to be injective we actually require the columns of the matrix of the linear map $d\mathbf{X}_r$ to have linearly independent columns. One simple way to check if this condition is satisfied or not is to see $\det(d\mathbf{X}_r^T d\mathbf{X}_r) \neq 0$.

1.2 Examples

Graphs : Consider $f : \mathbb{R}^2 \to \mathbb{R}$ and a surface S, such that.

$$S = \{(x, y, f(x, y)) : (x, y) \in U \subseteq \mathbb{R}^2\}. \tag{2}$$

Here if f has smooth partial derivatives of all orders and is well defined over U, S satisfies the conditions required for it to be a regular surface. Simple check for the condition **(iii)** is that the two columns of $d\mathbf{X}_r$ are

$$\begin{bmatrix} 1 & 0 & \frac{\partial f}{\partial x} \end{bmatrix}^T , \begin{bmatrix} 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix}^T \tag{3}$$

They are linearly independent for any smooth function f so we have a regular surface S. *Unit Sphere* : $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$. We first verify that the map $\mathbf{X}_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$\mathbf{X}_1(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U \tag{4}$$

Where $U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ is a parametrization of S^2. Observe that $\mathbf{X}_1(U)$ is the open part of S^2 above the xy plane. Since $x^2 + y^2 < 1$, the function $+\sqrt{1 - (x^2 + y^2)}$ has continuous partial derivatives of all orders in U. Hence $\mathbf{X}_1$ satisfies the condition **(i)**. Condition **(iii)** can be easily verified by taking partial derivatives of $\mathbf{X}_1$ with respect to x and y, as they will be linearly independent (same argument as the one made for the graphs). To check the condition **(ii)**, we observe that $\mathbf{X}_1$ is one to one and that $\mathbf{X}_1^{-1}$ is the restriction of the continuous projection $\pi(x, y, z) = (x, y)$ to the set $\mathbf{X}_1(U)$. Hence $\mathbf{X}_1^{-1}$ is continuous in $\mathbf{X}_1(U)$. Similarly we can define $\mathbf{X}_2(x, y) = (x, y, -\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U$. We can easily check that $\mathbf{X}_1(U) \cup \mathbf{X}_2(U)$ covers S^2 minus the equator. Then, using the xz and zy planes, we can define the parametrizations,

$$\begin{aligned} \mathbf{X}_3(x, z) &= (x, \sqrt{1 - (x^2 + z^2)}, z) \\ \mathbf{X}_4(x, z) &= (x, -\sqrt{1 - (x^2 + z^2)}, z) \\ \mathbf{X}_5(y, z) &= (\sqrt{1 - (y^2 + z^2)}, y, z) \\ \mathbf{X}_6(y, z) &= (-\sqrt{1 - (y^2 + z^2)}, y, z) \end{aligned} \tag{5}$$

Together all these 6 parametric cover S^2 completely.

Unit Sphere, Polar coordinates:
Let $V = \{(\theta, \phi); 0 < \theta < \pi, 0 < \phi < 2\pi\}$ and $\mathbf{X}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Clearly, $\mathbf{X}(V) \subset S^2$. Note that this map doesn't include the poles. If we include the poles the mapping $\mathbf{X}$ doesn't satisfy the condition **(iii)** at all points, to be precise at the poles. As far as the poles are not included the mapping $\mathbf{X}$ is a parametrization of S^2.
Note: No completely closed surface can be represented by a single regular parametric map.

Regular Value or Implicit representation: Next, suppose that S is given to us in terms of the zero level of a function $f(x, y, z)$. For $S = f^{-1}(0)$ to be a regular surface, f has to satisfies following conditions:

(i) $f : \mathbf{U} \subset \mathbb{R}^3 \to \mathbb{R}$ is a *smooth* function.

(ii) f_x, f_y and f_z do not vanish simultaneously at any point in the inverse image $f^{-1}(0) = \{(x, y, z) \in \mathbf{U} : f(x, y, z) = 0\}$.

Here f_x, f_y and f_z are partial derivatives with respect to x, y and z.

The ellipsoid : Is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ a regular surface? Let's take a function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ such that,

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \tag{6}$$

It is a smooth functions. The partial derivatives $f_x = 2x/a^2$, $f_y = 2y/a^2$ and $f_z = 2z/a^2$ vanish simultaneously only at $(0, 0, 0)$.

$$f^{-1}(0) = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\} \tag{7}$$

It is clear that $(0, 0, 0) \notin f^{-1}(0)$. So there doesn't exist a single point in $f^{-1}(0)$ such that f_x, f_y and f_z vanish simultaneously. So $f^{-1}(0)$ is a regular surface and 0 is referred to as a regular value of f.

2 The Tangent Plane

Before we get to define the tangent plane for a given regular surface S with a parametrization $\mathbf{X} : U \subseteq \mathbb{R}^2 \to V \cap S \subset \mathbb{R}^3$, let's define a couple of concepts required.

- Curves on a surface: Let's say we are given a parametrization of a regular curve γ such that,

$$\gamma : (-\epsilon, \epsilon) \to \mathbf{X}(U) \subset S, \quad \gamma(t) \in S, \forall t \in (-\epsilon, \epsilon) \tag{8}$$

  Based on the property of the regular curve and the regular surface we can define $\gamma_0 = \mathbf{X}^{-1} \circ \gamma : (-\epsilon, \epsilon) \to U$ and it is easy to $\gamma(t) = \mathbf{X}(\gamma_0(t)) \in S \ \ , \forall t \in (-\epsilon, \epsilon)$.

- Tangent vectors to a surface: Similarly let's say we are given a tangent vector w at $\mathbf{X}(q)$, that is , let $w = \gamma'(0)$. γ is defined as above. We know that γ_0 is differentiable, hence we have $d\mathbf{X}_q(\gamma_0'(0)) = w$. Note that $d\mathbf{X}_q$ is a linear map at point $q \in S$

Now we are ready to define the tangent plane at a point q on the surface S with the help of a parametric map $\mathbf{X}$.
Let $\mathbf{X} : U \subseteq \mathbb{R}^2 \to V \cap S \subset \mathbb{R}^3$ be a parametrization of a subset of a regular surface S. Let $q \in S$ such that, $q = \mathbf{X}(u)$ for some $u \in U$.
The tangent plane T_qS is defined as an $Image(d\mathbf{X}_q) \subseteq T_{\mathbf{X}(u)}\mathbb{R}^3$. Although we have defined the tangent plane based on the parametric map, it is important to note that the tangent plane doesn't depend upon the parametric map. At the same time the choice of parametrization $\mathbf{X}$ can be used to determine the basis $\{\frac{\partial \mathbf{X}}{\partial u}(q), \frac{\partial \mathbf{X}}{\partial v}(q)\}$ of the tangent plane. Where $\mathbf{X}(u, v) = q \in S$.