CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 7: The Second Fundamental Form

The unit normal vector of a surface.

- The normal vector of a surface. Is this geometric?
- The normal line is geometric but the normal direction may not be. Non-orientable surfaces.
- Normal vector of a parametrized surface, graph and level set.

Surface Area.

- Setting up the Riemann sum that yields the surface area of a surface.
- Area of infinitesimal coordinate rectangle and the Riemannian area form.
- Independence of parametrization of the area integral.

The Gauss map.

- Let S be an orientable surface with unit normal vector field n_p at each $p \in S$. The Gauss map of S is the mapping $N : S \to \mathbb{S}^2$ given by $N(p) := n_p$. Here we view the unit normal vector at a given $p \in S$ as a vector in $\mathbb{R}^3$ of length one and thus a point in $\mathbb{S}^2$.
- The Gauss map of a differentiable surface is itself differentiable. Thus we can study its differential $DN_p : T_pS \to T_{n_p}\mathbb{S}^2$.
  - We can define the differential rigorously as follows. Let V_p be a tangent vector to S at p generated by a curve $c : (-\varepsilon, \varepsilon) \to S$. In other words, $c(0) = p$ and $\frac{dc}{dt}\big|_{t=0} = V_p$. Then $DN_p(V_p) := \frac{d}{dt} N_p(c(t))\big|_{t=0}$. This is well-defined because we can show that the choice of curve doesn't matter.
  - Note that because $N(p) \in \mathbb{S}^2$ for each p it really is the case that $DN_p(V)$ is tangent to $\mathbb{S}^2$ for any vector $V \in T_pS$.
  - Prove this by differentiating $\|N(c(t))\|^2 = 1$ where $c : [-1, 1] \to S$ is a curve in S.
- Some examples. Gauss map of parametrized surface, level set and graph.

Definition of the second fundamental form.

- Since T_pS and $T_{n(p)}\mathbb{S}^2$ are parallel planes (they're both perpendicular to n_p), we can consider the differential of the Gauss map as a map $DN_p : T_pS \to T_pS$.
- Proposition: viewed in this way, DN_p is self-adjoint with respect to the Euclidean metric of $\mathbb{R}^3$ restricted to T_pS.
- Definition: the second fundamental form at $p \in S$ is the bilinear form $A_p : T_pS \times T_pS \to \mathbb{R}$ defined by $A_p(V, W) := -\langle DN_p(V), W\rangle$ for any $V, W \in T_pS$.
- $A_p(V, W)$ measure the projection onto W of the rate of change of N_p in the V-direction.
- Example calculations.

The second fundamental form as extrinsic curvature.

- Let $c : [-1, 1] \to S$ be a curve in S with $c(0) = p$. Then the geodesic curvature vector of c at zero is related to the second fundamental form at p as follows: $\langle \vec{k}_c(0), n_p \rangle = A_p(\dot{c}(0), \dot{c}(0))$. Note this is independent of $\ddot{c}$ or $c(t), \dot{c}(t)$ for $t \neq 0$.
- Let V vary over all unit vectors in T_pS. Then $A_p(V, V)$ takes on a minimum value k_{min} and a maximum value k_{max}. These are the *principal curvatures* of S at p and are eigenvalues of A_p. The corresponding eigenvectors V_{min} and V_{max} are the *principal directions* of A_p. Note that V_{min} and V_{max} are orthogonal.
- Mean curvature and Gauss curvature.
- Example calculations.

Local "shape" of a surface.

- Definitions of *elliptic*, *hyperbolic*, *parabolic*, *planar* or *umbilic* points.
- Examples of each type.
- Local characterization of the surface S at p depending its type. Proof based on Taylor series expansion in the "right" coordinate system: a neighbourhood of p is the graph of a function over T_pS.

Interpretation of the Gauss curvature in terms of the Gauss map.

- Lemma: $K(p) > 0$ iff Gauss map locally preserves orientation; $K(p) < 0$ iff Gauss map locally reverses orientation.
- Proposition: Let $p \in S$ be such that $K(p) \neq 0$ and let $\varepsilon > 0$ be such that K does not change sign in $B_\varepsilon(p)$. Then if N denotes the Gauss map, we have

$$K(p) = \lim_{\varepsilon \to 0} \frac{Area\big(N(B_\varepsilon(p))\big)}{Area(B_\varepsilon(p))}$$

Interpretation of the mean curvature as first variation of area.

- Another interpretation for the second fundamental form — at least of its trace, the mean curvature.
- The calculation.