CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 9: The Intrinsic Geometry of Surfaces

Reminder: the tangent plane.

- Recall the definition of the tangent plane.
- A parametrization gives you a basis for the tangent plane. The basis depends on the coordinates chosen; but the tangent plane itself does not.
- Vector transformation formulas.

The induced metric.

- The induced metric of a surface is very simply the restriction of the ambient Euclidean metric to each tangent space. You can now measure lengths of tangent vectors and angles between tangent vectors.
- A parametrization gives you a *representation* of the intrinsic metric in the parameter plane as a matrix.
- Different coordinates give different representations — but the underlying object is unchanged.
- So we have *transformation formulas* for the induced metric between representations.
- Examples: plane and cylinder, helicoid and catenoid, graphs.

Geodesics.

- Reminder of the relevant theory from the curves lecture.
- Geodesics on a surface.
- Length-minimizing properties of geodesics (local and global). Cut locus.
- Length-minimization and energy minimization.
- Equation of geodesics derived by first variation of the energy. (Not the equations involving the Christoffel symbols — just the fact that a geodesic c must have the arc length parametrization and also $[\ddot{c}(t)]^{\|} = 0$ or equivalently $\vec{k}_{c(t)} \perp T_{c(t)}\Sigma$.)
- Surface viewed as a metric space with an intrinsic notion of distance.

Geodesic normal coordinates.

- Definition of the exponential map $\exp_p : T_pM \to M$. Proof that $(D\exp_p)_0 = id$ so that $\exp_p$ is a diffeomorphism near the origin in T_pM.
- Gauss Lemma: $\langle (D\exp_p)_v(v), (D\exp_p)_v(w)\rangle = g_p(v, w)$.
- Define the *geodesic normal coordinates*. Form of the metric in these coordinates.
- Contrast with *isothermal coordinates*.

Surface Area.

- Setting up the Riemann sum that yields the surface area of a surface.
- Area of infinitesimal coordinate rectangle and the Riemannian area form.
- Independence of parametrization of the area integral.
- First variation of area.